Problem Set 0

Due: Thursday Jan 29 at 12:30 in class (i.e., SC 1105)

This homework contains four problems (and one extra credit problem). Please follow the homework format guidelines posted on the class web page:

http://www.cs.uiuc.edu/class/sp09/cs373/

In particular, submit each problem on a **separate sheet of paper**, put your name on each sheet, and write your discussion section time and day (e.g. Tuesday 10am) in the upper righthand corner. These details may sound picky, but they make the huge pile of homeworks much easier to grade quickly and more importantly, since we return them in the discussion sections, easier for you to get them back.

Note, that this homework should be done on your own. The next homeworks in this class could be done will working in groups. See webpage for details.

1. We are all individuals.

[Category: Proof, Points: 10]

The following is an inductive proof of the following statement:¹

"All students enrolled in CS373 in Spring 2009 have hair of the same color" (Which I hope you realize is not true.).

Proof: For any subset S of students enrolled, we will argue that their hair is of the same color. The proof is by induction on the size of S.

Base case: Base case is when |S| = 1. A set containing only one student clearly satisfies the statement.

Inductive step: Assume the statement is true for all subsets of students S', where |S'| = k. We will show that the statement is true for all subsets of students S, where |S| = k + 1.

Let |S| = k + 1. Remove one student, say x, from S to get a set $S_{\backslash x}$. By the inductive assumption, all students in $S_{\backslash x}$ have hair of the same color. Now remove another student y from S to get a set $S_{\backslash y}$; again, all students in $S_{\backslash y}$ have hair of the same color. Now, since x's color agrees with the rest of the students in the set $S_{\backslash y}$ and y's color agrees with the rest of the students in $S_{\backslash x}$, x and y must also have hair of the same color!

Let us illustrate the argument for a particular k, say 5. Assume $S = \{x, y, z, a, b\}$. Then $S \setminus \{x\} = \{y, z, a, b\}$ is a 4-element set, and hence the students in this set have the same hair color. Similarly $S \setminus \{y\} = \{x, z, a, b\}$ all have the same hair color. Hence, since x's hair color is the same as, say z, and y's color also agrees with z's, x and y

¹We will assume for the sake of simplicity that each student has hair, and this hair is of only one color!

have the same hair color, and hence the students in $S = \{x, y, z, a, b\}$ all have the same hair color.

Show why the above induction argument is wrong. In particular, illustrate one set for which the inductive argument fails.

2. Trees and edges.

[Category: Proof, Points: 10]

The following is an inductive proof of the statement that in every tree T = (V(T), E(T)), |E(T)| = |V(T)| - 1, i.e a tree with n vertices has n - 1 edges.

<u>Proof</u>: The proof is by induction on |V(T)|.

Base case: Base case is when |V(T)| = 1. But a tree with a single vertex has no edge, so |E(T)| = 0. Therefore in this case the formula is true since 0 = 1 - 1.

Inductive step: Assume that the formula is true for all trees T where |V(T)| = k. We will prove that the formula is true for trees with k + 1 nodes. A tree T with k + 1 nodes can be obtained from a tree T' with k nodes by attaching a new vertex to a leaf of T'. This way we add exactly one vertex and one edge to T', so |V(T)| = |V(T')| + 1 and |E(T)| = |E(T')| + 1. Since |V(T')| = k by induction hypothesis we have |E(T')| = |V(T')| = |V(T')| - 1.

Combining the last three relations we have |E(T)| = |E(T')| + 1 = |V(T')| - 1 + 1 = |V(T)| - 1 - 1 + 1 = |V(T)| - 1, which means that the formula is true for tree T.

Show that the above is *not* a correct inductive proof! You must argue why it is not correct, and in particular produce a tree which the above argument does not cover.

3. Number of leaves.

[Category: Proof, Points: 10]

Give a (correct) inductive proof of the following claim:

Let T be a full binary tree over n vertices (that is, every node in T other than the leaves has two children). Then T has (n+1)/2 leaves.

4. True/false/whatever.

[Category: Notation, Points: 20]

Answer each of the following with true, false or meaningless. The notation \setminus denotes "set-minus" or "set-difference".

- D1) $\emptyset \in \{\emptyset, 1\}$
- D2) $\varnothing \subseteq \{\varnothing, 1\}$
- D3) $1 \subseteq \{\emptyset, 1\}$
- D4) $\{1\} + \{1, 2\} = \{1, 2\}$
- D5) $\{1,2\} \setminus \{1\} = \{2\}$

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D6) \{1,2\} \setminus \{0\} = \{1,2\}

D7) \{1,2\} \cap \{3,4\} = \{\}

D8) \{1,2\} \cap \{3,4\} = \{\emptyset\}

D9) \{1,2\} \cup \{1,3\} = \{1,1,2,3\}

D10) \{1,2\} \cup \{1,3\} = \{1,2,3\}

D11) \{1,\{1\},\{\{1\}\}\} = \{1\}

D12) \{1,\{1\},\{\{1\}\}\} = \{1\}

D13) \{1\} \in \{1,\{1\},\{\{1\}\}\}\}

D14) \{1\} \subseteq \{1,\{1\},\{\{1\}\}\}\}

D15) \{\{1\}\} \in \{1,\{1\},\{\{1\}\}\}\}

D16) \{A,B\} \times \{C,D\} = \{(A,B),(C,D)\}.

D17) (\{A,B\} \times \{C,D\}) \cap (\{C,D\} \times \{A,B\}) = \{\}.

D18) |\{A,B,C\} \times \{D,E\}| = 6.

D19) \{A,B\} \times \{\} = \{A,B\}.

D20) \{A,B\} \setminus \{B,A\} = \{\}.
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5. Getting to 100.

[Category: Proof, Points: 10]

(Extra credit.)

We are given one copy of every digit in the list 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. You are asked to form numbers from these digits (you can use each digit only once, and you must use each digit once), such that the sum of these numbers add up to 100.

For example, a valid set of numbers you can form from the digits is $\{23, 17, 40, 5, 6, 8, 9\}$ but it adds up to 108, not 100.

Prove that this is impossible; i.e. no matter how you form the numbers, you *can not* find a way for them to add up to exactly 100.