Lecture #7

Nondeterministic finite automata

and

Deterministic finite automata

are equivalent

$$
\begin{tikzcd}
\mathrm{DFA} \arrow[r, bend left, "\text{trivial}"] & \mathrm{NFA} \arrow[l, "\text{today's lecture}"] & \mathrm{Regexp} \arrow[l, bend right, "\text{last lec.}"']
\end{tikzcd}
$$

$\mathrm{DFA} \hookrightarrow \mathrm{NFA}$

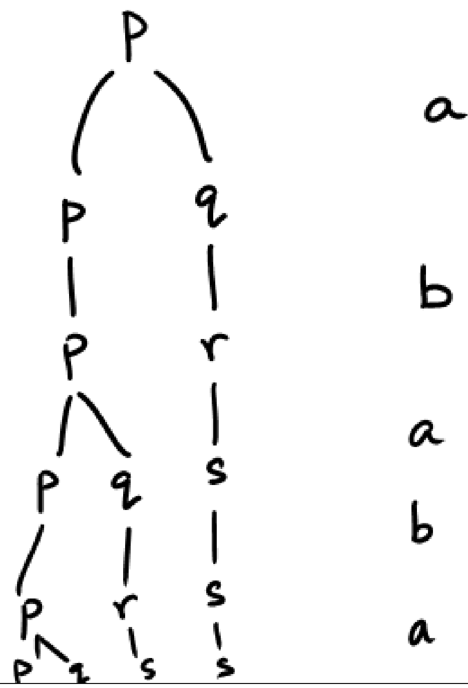
<u>Formally</u>. DFA $A = (Q, \Sigma, \delta, q_0, F)$

NFA $B = (Q, \Sigma, \delta', q_0, F)$

$\forall q \in Q, a \in \Sigma \;:\; \delta'(q, a) = \{\delta(q, a)\}$

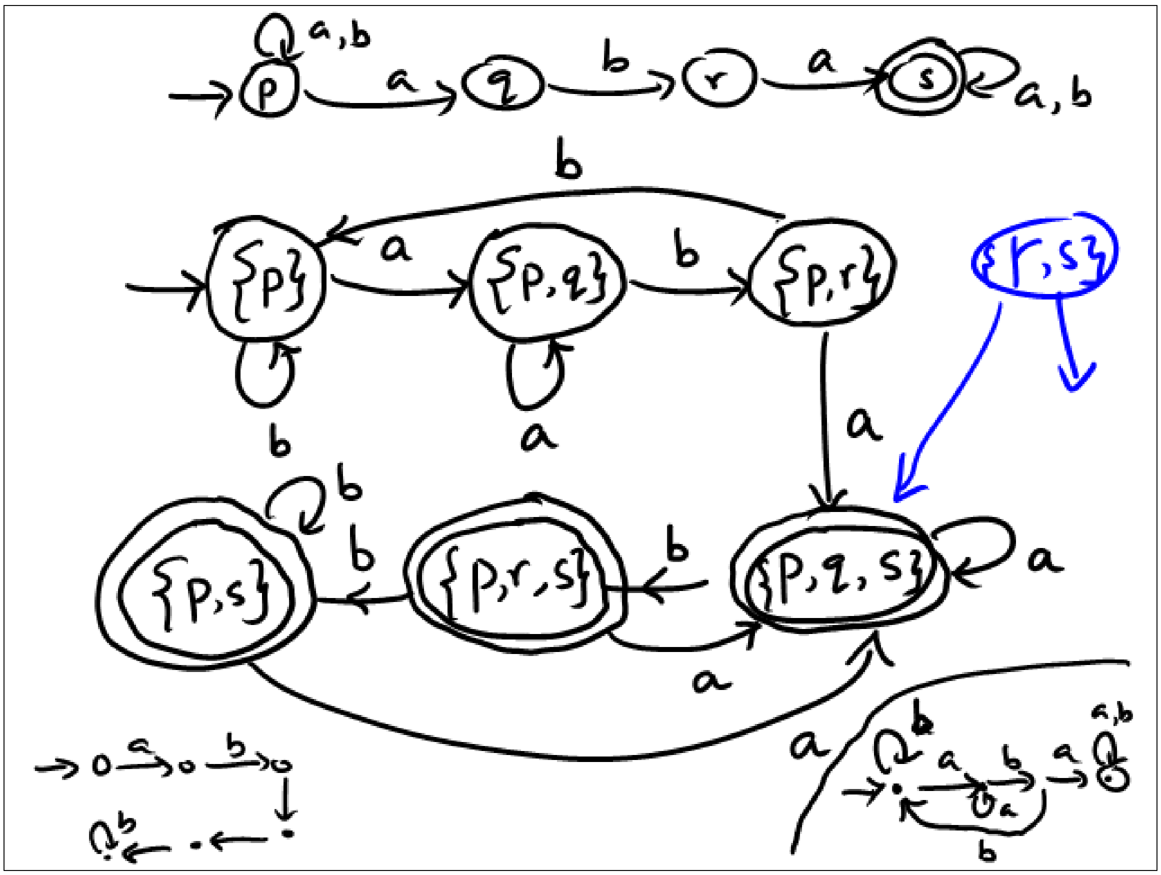
$\forall q \in Q \quad \delta'(q, \epsilon) = \emptyset$

ababa



What must a DFA remember after reading w?

The <u>precise</u> set of states the NFA can be in after reading w.



a,b
p
a
q
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r
a
s
a,b
b
{p}
a
{p,q}
b
{p,r}
{r,s}
b
a
a
b
b
{p,s}
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{p,r,s}
b
{p,q,s}
a
a
a
a,b

Lemma For any NFA without ε-transitions, there is a DFA that accepts the same language.

Construction. NFA $A = (Q, \Sigma, \delta, q_0, F)$

DFA $B = (\mathcal{P}(Q), \Sigma, \hat{\delta}, \{q_0\}, \hat{F})$

$$\hat{F} = \{ X \subseteq Q \mid X \cap F \neq \emptyset \}$$

(or $X \in \mathcal{P}(Q)$)

$$\hat{\delta}(X, a) = \bigcup_{q \in X} \delta(q, a) \qquad X \subseteq Q, a \in \Sigma$$

note: this is a single state of the DFA

Proof that the construction works.

Claim: For any $w \in \Sigma^*$, the set of states reached by NFA on w is precisely the state reached by DFA on w.

i.e. $\delta(q_0, w) = \hat{\delta}(\{q_0\}, w)$

Proof by induction on $|w|$

$|w| = 0$: $w = \epsilon$. NFA can reach only $\{q_0\}$
DFA on ϵ reaches $\{q_0\}$

$|w| = k+1$ $(k \geq 0)$, $w = w_1 a$.

By Ind hypo, set of states X reached by NFA on w_1 is equal to the stated by DFA on w_1

$$\delta(q_0, w) = \bigcup_{q \in X} \delta(q, a)$$

$$= \hat{\delta}(X, a) \quad \left(\text{since } \hat{\delta}(X,a) = \bigcup_{q\in X} \delta(q,a) \text{ by defn.}\right)$$

$$= \hat{\delta}(\{q_0\}, w_1 a) \quad \left(\text{since } \hat{\delta}(\{q_0\}, w_1) = X,\ \hat{\delta}(\{q_0\}, w_1 a) = \hat{\delta}(X,a)\right)$$

$$= \hat{\delta}(\{q_0\}, w) \quad (\text{since } w = w_1 a)$$

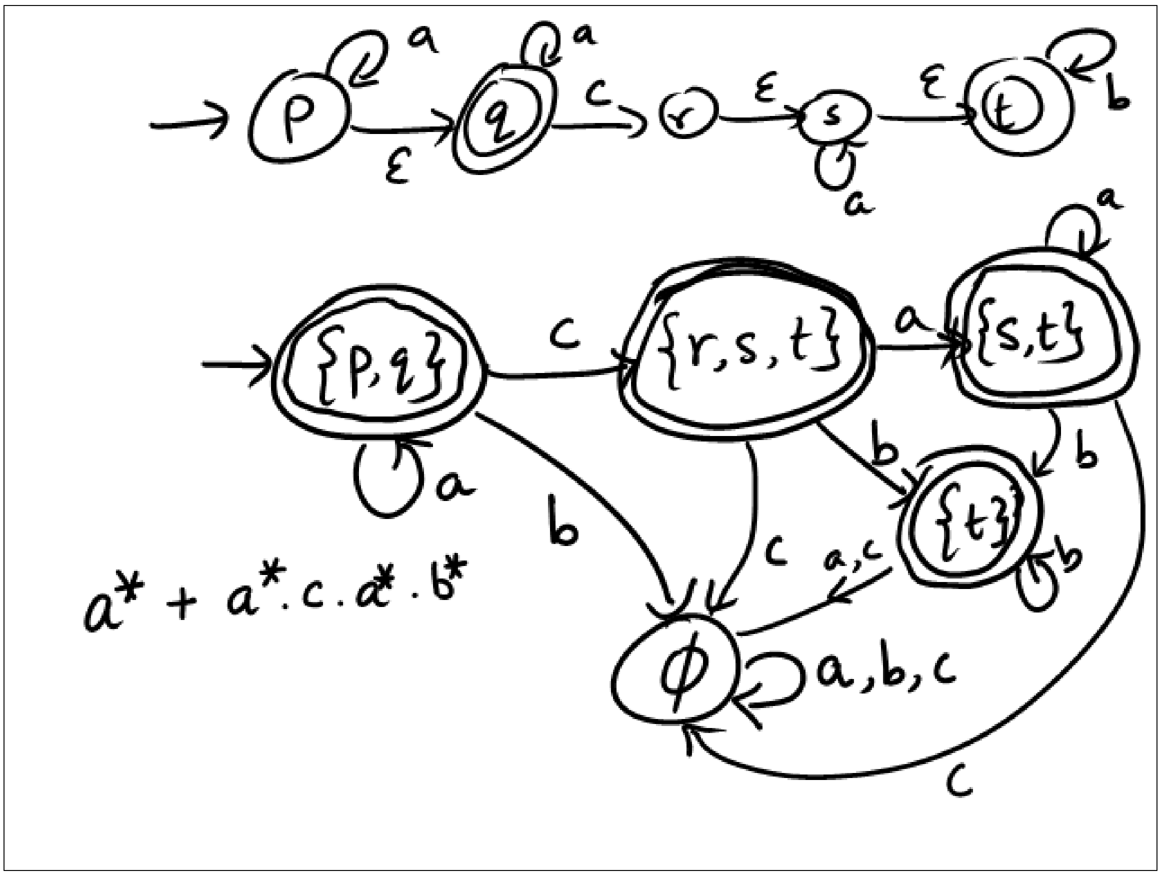
Now to show $L(A) = L(B)$

Let $w \in \Sigma^*$

$w \in L(A)$ iff $\underline{\delta(q_0, w)} \cap F \neq \emptyset$

iff $\hat{\delta}(\{q_0\}, w) \cap F \neq \emptyset$

iff $\hat{\delta}(\{q_0\}, w) \in \hat{F}$ iff $w \in L(B)$.



p
q
r
s
t
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ε
ε
ε
a
b
{p,q}
{r,s,t}
{s,t}
{t}
ϕ
c
a
a
a
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b
b
c
a,c
b
a,b,c
c
a* + a*.c.a*.b*

Formally, for any $X \subseteq Q$

$$E(X) = \{ q \mid q \text{ can be reached from some state in } X \text{ using 0 or more epsilon transitions} \}$$

$$X \subseteq E(X)$$

NFA: $A = (Q, \Sigma, \delta, q_0, F)$

with or without ϵ-trans.

DFA $B = (\mathcal{P}(Q), \Sigma, \hat{\delta}, \hat{q}_0, \hat{F})$

$\hat{q}_0 = E(\{q_0\})$; $\hat{F} = \{X \subseteq Q \mid X \cap F \neq \emptyset\}$

$$\hat{\delta}(X, a) = E\left(\bigcup_{q \in X} \delta(q, a)\right)$$