Lecture #4

- Formalizing the product construction
- Regular expressions

Formalizing the product construction.

$$M = (Q, \Sigma, \delta, q_0, F)$$

$$M' = (Q', \Sigma, \delta', q_0', F')$$

A product automaton of M_1 and M_2 is

$$N = (Q \times Q', \Sigma, \delta_N, (q_0, q_0'), F_N)$$

where $\forall (q, q') \in Q \times Q'$, $a \in \Sigma$

$$\delta_N((q, q'), a) = (\delta(q, a), \delta'(q', a))$$

$$F_N \subseteq (Q \times Q')$$

- Intersection automaton of M and M' is the product automaton of M and M':

$$N = (Q \times Q', \Sigma, \delta_N, (q_0, q_0'), F_N)$$

where $F_N = F \times F'$

- Union automaton of M and M' is where $F_N = (F \times Q') \cup (Q \times F')$

More notation

$$M = (Q, \Sigma, \delta, q_0, F)$$

$$\delta : Q \times \Sigma \longrightarrow Q$$

Extend this function: $\delta : Q \times \Sigma^* \longrightarrow Q$

$$\delta(q, \epsilon) = q \quad \forall q \in Q$$

$$\delta(q, wa) = \delta(\delta(q, w), a)$$

M accepts w iff $\delta(q_0, w) \in F$

Back to the product construction.

Let $M = (Q, \Sigma, \delta, q_0, F)$

$M' = (Q', \Sigma, \delta', q_0', F')$

and let N be a product automaton

$$N = (Q \times Q', \Sigma, \delta_N, (q_0, q_0'), F_N)$$

where $\delta_N((q, q'), a) = (\delta(q, a), \delta'(q', a))$

$\forall q, q' \in Q, a \in \Sigma$.

<u>Lemma</u> For every $w \in \Sigma^*$

$$\delta_N((q_0, q_0'), w) = (\delta(q_0, w), \delta'(q_0', w))$$

Proof. (by induction on $|w|$)

Base case $|w| = 0$; $w = \epsilon$

$$\delta_N((q_0, q_0'), \epsilon) = (q_0, q_0')$$
$$= (\delta(q_0, \epsilon), \delta'(q_0', \epsilon))$$

Induction step $|w| > 0$, Let $w = w'a$

$$\delta_N((q_0, q_0'), w) = \delta_N((q_0, q_0'), w'a)$$
$$= \delta_N(\delta_N((q_0, q_0'), w'), a)$$
$$= \delta_N((\underline{\delta(q_0, w')}, \delta'(\underline{q_0'}, w')), a)$$

by ind hypo

$$= \left(\delta(\delta(q_0, w'), a),\ \delta'(\delta'(q_0', w'), a)\right)$$

$$= \left(\delta(q_0, w'a),\ \delta'(q_0', w'a)\right)$$

$$= \left(\delta(q_0, w),\ \delta'(q_0', w)\right)$$

$$\prime$$

Let $M = (Q, \Sigma, \delta, q_0, F)$

$M' = (Q', \Sigma, \delta', q_0', F')$

$\&$ $N = (Q \times Q', \delta_N (q_0, q_0'), F \times F')$

where $\delta_N((q, q'), a) = (\delta(q, a), \delta'(q', a))$

$\mathcal{L}(N) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$

Let $w \in \Sigma^*$. $w \in L(N)$

iff $\delta_N((q_0, q_0'), w) \in F \times F'$

iff $(\delta(q_0, w), \delta'(q_0', w)) \in F \times F'$

iff $\delta(q_0, w) \in F$ and $\delta'(q_0', w) \in F'$

iff $w \in L(M)$ and $w \in \mathcal{L}(M')$

iff $w \in L(M) \cap L(M')$

Three operations of interest:

$L_1 \cup L_2$ union

$L_1 \cdot L_2$ concatenation

L^* recursion.

$$L_1 \cdot L_2 = \{ w_1 . w_2 \mid w_1 \in L_1 , w_2 \in L_2 \}$$

Eg. $L_1 = \{ a, b \}$; $L_2 = \{ bc, ca \}$

$L_1 . L_2 = \{ abc, aca, bbc, bca \}$.

$L_1 \cdot L_2 = \{ w \mid \exists w_1, w_2 : w = w_1 \cdot w_2 , w_1 \in L_1 , w_2 \in L_2 \}$

$*$ – Kleene-$*$

$$L^* = \{ w \mid w = w_1 \ldots w_k \text{ where } k \geq 0 \text{ and each } w_i \in L \}$$

Eg. $\{a, bb\}^* = \{\epsilon, a, bb, abb, bba, aa, bbbb, abba, abbbb, \ldots\}$

$\{a,b\}^*$ = set of all words with letters a, b

This is why we call Σ^* the set of all words over Σ.

Regular expressions

$\forall a \in \Sigma$ $\quad a$ is a regexp. $\mapsto L(a) = \{a\}$

ϵ is a regexp $\mapsto L(\epsilon) = \{\epsilon\}$

ϕ is a regexp $\mapsto L(\phi) = \phi$

Combine regular expressions

$R+S$, $R \cup S$ is a regexp $\mapsto L(R \cup S) = L(R) \cup L(S)$

$R \cdot S$ is a regexp $\mapsto L(R.S) = L(R) \cdot L(S)$

R^* is a regexp $\mapsto L(R^*) = (L(R))^*$

How do I write Σ^* $\Sigma = \{a, b\}$

$(a \cup b)^*$

$(a + b)^*$

$(aa)^* = \{ w \in \{a^*\} \mid |w| \text{ is even} \}$

$= \{ \epsilon, aa, aaaa, \ldots \}$

$(a+b)^* aab (a+b)^* = \{ w \in \{a,b\}^* \mid w = w_1 aab w_2 \}$

$$(a+b)^{*}\ aab\ (a+b)^{\cancel{*)}\,*} = L_1$$

$$(a+b)\ bba\ (a+b)^{*} = L_2$$

$\Sigma = \{a,b\}$

$$(a+b)\cdot(a+b)^{*} = \Sigma^{*} \setminus \{\epsilon\}$$