1 Closure Properties

1.1 Decidable Languages

Boolean Operators

**Proposition 1.** Decidable languages are closed under union, intersection, and complementation.

**Proof.** Given TMs $M_1$, $M_2$ that decide languages $L_1$, and $L_2$

- A TM that decides $L_1 \cup L_2$: on input $x$, run $M_1$ and $M_2$ on $x$, and accept iff either accepts. (Similarly for intersection.)

- A TM that decides $\overline{L_1}$: On input $x$, run $M_1$ on $x$, and accept if $M_1$ rejects, and reject if $M_1$ accepts. □

Regular Operators

**Proposition 2.** Decidable languages are closed under concatenation and Kleene Closure.

**Proof.** Given TMs $M_1$ and $M_2$ that decide languages $L_1$ and $L_2$.

- A TM to decide $L_1L_2$: On input $x$, for each of the $|x|+1$ ways to divide $x$ as $yz$: run $M_1$ on $y$ and $M_2$ on $z$, and accept if both accept. Else reject.

- A TM to decide $L_1^*$: On input $x$, if $x = \epsilon$ accept. Else, for each of the $2^{|x|-1}$ ways to divide $x$ as $w_1 \ldots w_k$ ($w_i \neq \epsilon$): run $M_1$ on each $w_i$ and accept if $M_1$ accepts all. Else reject. □

Inverse Homomorphisms

**Proposition 3.** Decidable languages are closed under inverse homomorphisms.

**Proof.** Given TM $M_1$ that decides $L_1$, a TM to decide $h^{-1}(L_1)$ is: On input $x$, compute $h(x)$ and run $M_1$ on $h(x)$; accept iff $M_1$ accepts. □

Homomorphisms

**Proposition 4.** Decidable languages are not closed under homomorphism

**Proof.** We will show a decidable language $L$ and a homomorphism $h$ such that $h(L)$ is undecidable

- Let $L = \{xy \mid x \in \{0, 1\}^*, y \in \{a, b\}^*, x = \langle M, w \rangle$, and $y$ encodes an integer $n$ such that the TM $M$ on input $w$ will halt in $n$ steps \}

- $L$ is decidable: can simply simulate $M$ on input $w$ for $n$ steps

- Consider homomorphism $h$: $h(0) = 0$, $h(1) = 1$, $h(a) = h(b) = \epsilon$.

- $h(L) = \text{HALT}$ which is undecidable. □
1.2 Recursively Enumerable Languages

Boolean Operators

**Proposition 5.** R.E. languages are closed under union, and intersection.

*Proof.* Given TMs $M_1, M_2$ that recognize languages $L_1, L_2$

- A TM that recognizes $L_1 \cup L_2$: on input $x$, run $M_1$ and $M_2$ on $x$ in parallel, and accept iff either accepts. (Similarly for intersection; but no need for parallel simulation)

Complementation

**Proposition 6.** R.E. languages are not closed under complementation.

*Proof.* $A_{TM}$ is r.e. but $\overline{A_{TM}}$ is not.

Regular Operations

**Proposition 7.** R.E languages are closed under concatenation and Kleene closure.

*Proof.* Given TMs $M_1$ and $M_2$ recognizing $L_1$ and $L_2$

- A TM to recognize $L_1L_2$: On input $x$, do in parallel, for each of the $|x| + 1$ ways to divide $x$ as $yz$: run $M_1$ on $y$ and $M_2$ on $z$, and accept if both accept. Else reject.

- A TM to recognize $L_1^*$: On input $x$, if $x = \epsilon$ accept. Else, do in parallel, for each of the $2^{|x|-1}$ ways to divide $x$ as $w_1 \ldots w_k (w_i \neq \epsilon)$: run $M_1$ on each $w_i$ and accept if $M_1$ accepts all. Else reject.

Homomorphisms

**Proposition 8.** R.E. languages are closed under both inverse homomorphisms and homomorphisms.

*Proof.* Let TM $M_1$ recognize $L_1$.

- A TM to recognize $h^{-1}(L_1)$: On input $x$, compute $h(x)$ and run $M_1$ on $h(x)$; accept iff $M_1$ accepts.

- A TM to recognize $h(L_1)$: On input $x$, start going through all strings $w$, and if $h(w) = x$, start executing $M_1$ on $w$, using dovetailing to interleave with other executions of $M_1$. Accept if any of the executions accepts.