Problem 1. [Category: Proof] Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be an NFA recognizing language $L$. Recall that in lecture 7 we gave the following construction of an NFA recognizing $L^*$: Take $N = (Q, \Sigma, \delta, q_0, F)$ where

- $Q = Q_1 \cup \{q_0\}$, where $q_0 \notin Q_1$
- $F = F_1 \cup \{q_0\}$
- $\delta$ is defined as follows
  
  $$
  \delta(q, a) = \begin{cases}
  \delta_1(q, a) & \text{if } q \in (Q_1 \setminus F_1) \text{ or } a \neq \epsilon \\
  \delta_1(q, a) \cup \{q_1\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\
  \{q_1\} & \text{if } q = q_0 \text{ and } a = \epsilon \\
  \emptyset & \text{otherwise}
  \end{cases}
  $$

Prove that $N$ does indeed recognize $L^*$, i.e., $L(N) = (L(N_1))^* = L^*$. [10 points]

Problem 2. [Category: Proof] Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be DFAs recognizing $L_1$ and $L_2$, respectively. Recall that in lecture 8, we gave the following construction of a DFA recognizing the intersection of $L_1$ and $L_2$: $M = (Q, \Sigma, \delta, q_0, F)$ defined as follows

- $Q = Q_1 \times Q_2$
- $q_0 = (q_1, q_2)$
- $\delta((p_1, p_2), a) = (\delta_1(p_1, a), \delta_2(p_2, a))$
- $F = F_1 \times F_2$

Prove that $L(M) = L(M_1) \cap L(M_2)$. [10 points]

Problem 3. [Category: Comprehension+Design] For languages $A$ and $B$ (over alphabet $\Sigma$), let $\text{PerfectShuffle}(A, B)$ be the language

$$
\{w \in \Sigma^* \mid w = a_1b_1a_2b_2\cdots a_nb_n, \text{ where, for each } i, a_i, b_i \in \Sigma \text{ and } a_1a_2\cdots a_n \in A \text{ and } b_1b_2\cdots b_n \in B\}
$$

1. Let $\Sigma = \{0, 1\}$. Let $A = L(0^*)$ and $B = L(1^*)$. What is $\text{PerfectShuffle}(A, B)$? [1 points]
2. Let $\Sigma = \{0\}$. Let $C = \{w \in \Sigma^* \mid |w| \text{ is even}\}$ and $D = \{w \in \Sigma^* \mid |w| \text{ is prime}\}$. What is $\text{PerfectShuffle}(C, D)$? [1 points]

3. Prove that regular languages are closed under the perfect shuffle operator, i.e., if $A$ and $B$ are regular then $\text{PerfectShuffle}(A, B)$ is regular. If you prove regularity of $\text{PerfectShuffle}(A, B)$ by constructing a DFA/NFA/regular expression then you need not prove the correctness of your construction; you should, however, clearly explain the intuition behind your construction. [8 points].