1 Reductions

1.1 Definitions and Observations

Mapping Reductions

Definition 1. A function $f : \Sigma^* \to \Sigma^*$ is *computable* if there is some Turing Machine M that on every input w halts with $f(w)$ on the tape.

Definition 2. A *reduction* (a.k.a. mapping reduction/many-one reduction) from a language A to a language B is a computable function $f : \Sigma^* \to \Sigma^*$ such that

$$w \in A \text{ if and only if } f(w) \in B$$

In this case, we say A is *reducible* to B, and we denote it by $A \leq_m B$.

Convention
In this course, we will drop the adjective "mapping" or "many-one", and simply talk about reductions and reducibility.

Reductions and Recursive Enumerability

Proposition 3. *If $A \leq_m B$ and B is r.e., then A is r.e.*

Proof. Let f be a reduction from A to B and let M_B be a Turing Machine recognizing B. Then the Turing machine recognizing A is

On input w
    Compute $f(w)$
    Run M_B on $f(w)$
    Accept if M_B accepts, and reject if M_B rejects □

Corollary 4. *If $A \leq_m B$ and A is not r.e., then B is not r.e.*

Reductions and Decidability

Proposition 5. *If $A \leq_m B$ and B is decidable, then A is decidable.*

Proof. Let f be a reduction from A to B and let M_B be a Turing Machine *deciding* B. Then a Turing machine that decides A is

On input w
    Compute $f(w)$
    Run M_B on $f(w)$
    Accept if M_B accepts, and reject if M_B rejects □

Corollary 6. *If $A \leq_m B$ and A is undecidable, then B is undecidable.*

1.2 Examples

The Halting Problem

Proposition 7. *The language HALT* $= \{\langle M, w\rangle \mid M$ *halts on input* $w\}$ *is undecidable.*

Proof. Recall $A_{\text{TM}} = \{\langle M, w\rangle \mid w \in L(M)\}$ is undecidable. Will give reduction f to show $A_{\text{TM}} \leq_m$ HALT $\implies$ HALT undecidable.

Let $f(\langle M, w\rangle) = \langle N, w\rangle$ where N is a TM that behaves as follows:

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On input x
    Run M on x
    If M accepts then halt and accept
    If M rejects then go into an infinite loop
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N halts on input w if and only if M accepts w. i.e., $\langle M, w\rangle \in A_{\text{TM}}$ iff $f(\langle M, w\rangle) \in$ HALT □

Emptiness of Turing Machines

Proposition 8. *The language* $E_{\text{TM}} = \{\langle M\rangle \mid \mathbf{L}(M) = \emptyset\}$ *is not r.e.*

Proof. Recall $L_d = \{\langle M\rangle \mid M \notin \mathbf{L}(M)\}$ is not r.e.

L_d is reducible to E_{TM} as follows. Let $f(M) = \langle N\rangle$ where N is a TM that behaves as follows:

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On input x
    Run M on ⟨M⟩ for |x| steps
    Accept x only if M accepts ⟨M⟩ within |x| steps
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Observe that $\mathbf{L}(N) = \emptyset$ if and only if M does not accept $\langle M\rangle$ if and only if $\langle M\rangle \in L_d$. □

Checking Regularity

Proposition 9. *The language REGULAR* $= \{\langle M\rangle \mid \mathbf{L}(M)$ *is regular*$\}$ *is undecidable.*

Proof. We give a reduction f from A_{TM} to REGULAR. Let $f(\langle M, w\rangle) = \langle N\rangle$, where N is a TM that works as follows:

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On input x
    If x is of the form 0^n 1^n then accept x
    else run M on w and accept x only if M does
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If $w \in \mathbf{L}(M)$ then $\mathbf{L}(N) = \Sigma^*$. If $w \notin \mathbf{L}(M)$ then $\mathbf{L}(N) = \{0^n1^n \mid n \geq 0\}$. Thus, $\langle N\rangle \in$ REGULAR if and only if $\langle M, w\rangle \in A_{\text{TM}}$ □

Checking Equality

Proposition 10. $EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid \mathbf{L}(M_1) = \mathbf{L}(M_2)\}$ *is not r.e.*

Proof. We will give a reduction f from E_{TM} to EQ_{TM}. Let M_1 be the Turing machine that on any input, halts and rejects i.e., $\mathbf{L}(M_1) = \emptyset$. Take $f(M) = \langle M, M_1\rangle$.

Observe $\langle M\rangle \in E_{\text{TM}}$ iff $\mathbf{L}(M) = \emptyset$ iff $\mathbf{L}(M) = \mathbf{L}(M_1)$ iff $\langle M, M_1\rangle \in \text{EQ}_{\text{TM}}$. □
