1 High-Level Descriptions of Computation

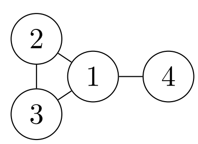
High-Level Descriptions of Computation

- Instead of giving a Turing Machine, we shall often describe a program as code in some programming language (or often "pseudo-code")
  - Possibly using high level data structures and subroutines
- Inputs and outputs are complex objects, encoded as strings
- Examples of objects:
  - Matrices, graphs, geometric shapes, images, videos, ...
  - DFAs, NFAs, Turing Machines, Algorithms, other machines ...

Encoding Complex Objects

- "Everything" finite can be encoded as a (finite) string of symbols from a finite alphabet (e.g. ASCII)
  - Can in turn be encoded in binary (as modern day computers do). No special $\sqcup$ symbol: use self-terminating representations

Example 1. A "graph" can be encoded as $\langle (1,2,3,4)((1,2)(2,3)(3,1)(1,4)) \rangle$ where the graph is



Notation
For any object O, we will use $\langle O \rangle$ to denote its representation as a binary string.

- Thus, if M is a DFA/PDA/TM then $\langle M \rangle$ is its encoding as a binary string.
- If G is a graph then $\langle G \rangle$ is its representation as a string.
- If $O_1, O_2, \ldots O_n$ are objects then $\langle O_1, \ldots O_n \rangle$ is the representation of these objects as a single string.

Problems with Programs/Machines as Input

- We will often consider problems where machines/programs are given as input.
  - Given an NFA, construct the equivalent DFA; given an NFA N and word w, decide if $w \in \mathbf{L}(N)$; ...

- All of these algorithms can be implemented on a Turing machine
- Some of these algorithms are for decision problems, while others are for computing more general functions

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Decision Problems and Languages

Recall

- Decision problems are problems that require a yes/no answer on a given input
- They have an exact correspondence to languages: L is a representation of problem P if and only if an input $x \in L$ iff answer for x is yes in problem P.

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2 Deciding vs. Recognizing

Decidable and Recognizable Languages

Recognizable Language
A Turing machine M *recognizes* language L if $L = \mathbf{L}(M)$. We say L is *Turing-recognizable* (or simply recognizable) if there is a TM M such that $L = \mathbf{L}(M)$.

Decidable Language
A Turing machine M *decides* language L if $L = \mathbf{L}(M)$ *and* M *halts on all inputs*. We say L is *decidable* if there is a TM M that decides L.

Decidable Problems

The following problems are all decidable.

- **Problem:** Given a DFA M and input w decide if M accepts w. We can write this formally as a language (using our notation) as $A_{\text{DFA}} = \{\langle M, w\rangle \mid M \text{ is a DFA and } w \in \mathbf{L}(M)\}$.

  Algorithm: "Simulate" M on w and answer "yes" iff M reaches a final state.

- **Problem:** Given a NFA M and input w decide if M accepts w. We can write this formally as a language (using our notation) as $A_{\text{NFA}} = \{\langle M, w\rangle \mid M \text{ is an NFA and } w \in \mathbf{L}(M)\}$.

  Algorithm: Convert M into a DFA and run the algorithm for A_{DFA}.

- **Problem:** $A_{\text{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression and } w \in \mathbf{L}(R)\}$.

  Algorithm: Convert R into a NFA and run the algorithm for A_{NFA}.

- **Problem:** Given a DFA M answer "yes" iff $\mathbf{L}(M) = \emptyset$. Formally,

$$E_{\text{DFA}} = \{\langle M \rangle \mid M \text{ is a DFA s.t. } \mathbf{L}(M) = \emptyset\}$$

  Algorithm: Check if a final state is reachable from the start state by using a graph search algorithm like DFS/BFS.

- **Problem:** Given DFA A and B, check if $\mathbf{L}(A) = \mathbf{L}(B)$. In other words,

$$EQ_{\text{DFA}} = \{\langle A, B \rangle \mid A, B \text{ are DFAs s.t. } \mathbf{L}(A) = \mathbf{L}(B)\}.$$

  Algorithm: Construct (using cross-product construction) the DFA C recognizing $(\mathbf{L}(A) \cap \overline{\mathbf{L}(B)}) \cup (\overline{\mathbf{L}(A)} \cap \mathbf{L}(B))$ and check if $\mathbf{L}(C) = \emptyset$.

- **Problem:** $A_{\text{CFG}} = \{\langle G, w \rangle \mid G \text{ is a CFG s.t. } w \in \mathbf{L}(G)\}$.

  Algorithm: Convert G to G' in Chomsky normal form. Now $w \in \mathbf{L}(G')$ iff w can be derived in $2|w| - 1$ steps, where none of the intermediate strings is of length more than $|w|$. Go through all such derivations (which is finite) and check if they derive w.

2.1 An Undecidable but Recognizable Language

Decidable and Recognizable Languages

- But *not all languages are decidable*! In the next class we will see an example:
  - $A_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } w \in \mathbf{L}(M)\}$ is undecidable
- However A_{TM} is *Turing-recognizable*!

  Proposition 2. *There are languages which are recognizable, but not decidable*

Recognizing A_{TM}

Program U for *recognizing* A_{TM}:

```
On input ⟨M,w⟩
    simulate M on w
    if simulated M accepts w, then accept
    else reject (by moving to q_rej)
```

U (the Universal TM) accepts $\langle M, w \rangle$ iff M accepts w. i.e.,

$$\mathbf{L}(U) = A_{\text{TM}}$$

But U does not *decide* A_{TM}: If M rejects w by not halting, U rejects $\langle M, w \rangle$ by not halting. Indeed (as we shall see) no TM decides A_{TM}.

2.2 Complementation

Deciding vs. Recognizing

Proposition 3. *If L and $\overline{L}$ are recognizable, then L is decidable*

Proof. Program P for *deciding* L, given programs P_L and $P_{\overline{L}}$ for recognizing L and $\overline{L}$:

- On input x, simulate P_L and $P_{\overline{L}}$ on input x. Whether $x \in L$ or $x \notin L$, one of P_L and $P_{\overline{L}}$ will halt in finite number of steps.
- Which one to simulate first? Either could go on forever.
- On input x, simulate *in parallel* P_L and $P_{\overline{L}}$ on input x until either P_L or $P_{\overline{L}}$ accepts
- If P_L accepts, accept x and halt. If $P_{\overline{L}}$ accepts, reject x and halt.

In more detail, P works as follows:

```
On input x
for i = 1, 2, 3, ...
    simulate P_L on input x for i steps
    simulate P_L̄ on input x for i steps
    if either simulation accepts, break
if P_L accepted, accept x (and halt)
if P_L̄ accepted, reject x (and halt)
```

(Alternately, maintain configurations of P_L and $P_{\overline{L}}$, and in each iteration of the loop advance both their simulations by one step.) □

Deciding vs. Recognizing
So far:

- A_{TM} is undecidable (next lecture)
- But it is recognizable
- Is every language recognizable? *No!*

Proposition 4. *$\overline{A_{\text{TM}}}$ is unrecognizable*

Proof. If $\overline{A_{\text{TM}}}$ is recognizable, since A_{TM} is recognizable, the two languages will be decidable too! □

Note: Decidable languages are closed under complementation, but recognizable languages are not.

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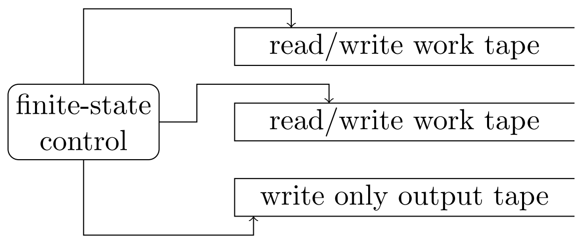
3 Recursive Enumeration

3.1 Enumerators

Enumerators



- An enumerator is multi-tape Turing Machine, with a special *output tape* which is *write-only*
  - Write-only means (a) symbol on output tape does not affect transitions, and (b) tape head only moves right.
- Intially all tapes blank (no input). During computation the machine adds symbols to the output tape. Output considered to be a *list of words* (separated by special symbol #)

Recursively Enumerable Languages

Definition 5. An enumerator M is said to *enumerate* a string w if and only if at some point M writes a word w on the output tape. $\mathbf{E}(M) = \{w \mid M \text{ enumerates } w\}$

Note
M need not enumerate strings in order. It is also possible that M lists some strings many times!

Definition 6. L is *recursively enumerable (r.e.)* iff there is an enumerator M such that $L = \mathbf{E}(M)$.

3.2 Equivalence of Enumerating and Recognizing a Language

Recursively Enumerable Languages and TMs

Theorem 7. *L is recursively enumerable if and only if L is Turing-recognizable.*

Note
Hence, when we say a language L is recursively enumerable (r.e.) then

- there is a TM that accepts L, and
- there is an enumerator that enumerates L.

Proof. **Enumerator to Recognizer:** Suppose L is enumerated by N. Need to construct M such that $\mathbf{L}(M) = \mathbf{E}(N)$. M is the following TM

```
On input w
    Run N.  Every time N writes a word 'x'
    compare x with w.
    If x = w then accept and halt
    else continue simulating N
```

Clearly, if $w \in L$, M accepts w, and if $w \notin L$ then M never halts.

Flawed Solution to Construct an enumerator: Let M be such that $L = \mathbf{L}(M)$. Need to construct N such that $\mathbf{E}(N) = \mathbf{L}(M)$. N is the following enumerator

```
for w = ε, 0, 1, 00, 01, 10, 11, 000, ... do
    simulate M on w
    if M accepts w then write the word 'w'
        on output tape
```

Does N enumerate L? *No!!* M may not halt on a string $w \notin L$, in which case N will not output any more strings! Therefore, one must simulate M on all inputs in parallel. But that means we need to have infinitely many parallel executions. How can this be accomplished?

Correct Construction using Dovetailing: Let M be such that $L = \mathbf{L}(M)$. Need to construct N such that $\mathbf{E}(N) = \mathbf{L}(M)$. N is the following enumerator

```
for i = 1, 2, 3 ... do
    let w_1, w_2, ... w_i be the first i strings (in
        lexicographic order)
    simulate M on w_1 for i steps, then on w_2 for i
        steps and ...simulate M on w_i for i steps
    if M accepts w_j within i steps then write w_j
        (with separator) on output tape
```

Observe that $w \in \mathbf{L}(M)$ iff N will enumerates w. N will enumerate strings many times! □