1 Closure Properties

1.1 Homomorphisms

Homomorphism

Proposition 1. *Context free languages are closed under homomorphisms.*

Proof. Let $G = (V, \Sigma, R, S)$ be the grammar generating L, and let $h : \Sigma^* \to \Gamma^*$ be a homomorphism. A grammar $G' = (V', \Gamma, R', S')$ for generating $h(L)$:

- Include all variables from G (i.e., $V' \supseteq V$), and let $S' = S$
- Treat terminals in G as variables. i.e., for every $a \in \Sigma$
  - Add a new variable X_a to V'
  - In each rule of G, if a appears in the RHS, replace it by X_a
- For each X_a, add the rule $X_a \to h(a)$

G' generates $h(L)$. *(Exercise!)* □

Example 2. Let G have the rules $S \to 0S0|1S1|\epsilon$.
Consider the homorphism $h : \{0, 1\}^* \to \{a, b\}^*$ given by $h(0) = aba$ and $h(1) = bb$.
Rules of G' s.t. $\mathbf{L}(G') = \mathbf{L}(L(G))$:

$$\begin{aligned} S &\to X_0SX_0|X_1SX_1|\epsilon \\ X_0 &\to aba \\ X_1 &\to bb \end{aligned}$$

1.2 Inverse Homomorphisms

Inverse Homomorphisms

Recall: For a homomorphism h, $h^{-1}(L) = \{w \mid h(w) \in L\}$

Proposition 3. *If L is a CFL then $h^{-1}(L)$ is a CFL*

Proof Idea

For regular language L: the DFA for $h^{-1}(L)$ on reading a symbol a, simulated the DFA for L on $h(a)$. Can we do the same with PDAs?

- Key idea: store $h(a)$ in a "buffer" and process symbols from $h(a)$ one at a time (according to the transition function of the original PDA), and the next input symbol is processed only after the "buffer" has been emptied.
- Where to store this "buffer"? In the state of the new PDA!

Proof. Let $P = (Q, \Delta, \Gamma, \delta, q_0, F)$ be a PDA such that $\mathbf{L}(P) = L$. Let $h : \Sigma^* \to \Delta^*$ be a homomorphism such that $n = \max_{a \in \Sigma} |h(a)|$, i.e., every symbol of Σ is mapped to a string under h of length at most n. Consider the PDA $P' = (Q', \Sigma, \Gamma, \delta', q_0', F')$ where

- $Q' = Q \times \Delta^{\leq n}$, where $\Delta^{\leq n}$ is the collection of all strings of length at most n over Δ.
- $q_0' = (q_0, \epsilon)$
- $F' = F \times \{\epsilon\}$
- δ' is given by

$$\delta'((q, v), x, a) = \begin{cases} \{((q, h(x)), \epsilon)\} & \text{if } v = a = \epsilon \\ \{((p, u), b) \mid (p, b) \in \delta(q, y, a)\} & \text{if } v = yu,\ x = \epsilon, \text{ and } y \in \Delta \end{cases}$$

  and $\delta'(\cdot) = \emptyset$ in all other cases.

We can show by induction that for every $w \in \Sigma^*$

$$\langle q_0', \epsilon \rangle \xrightarrow{w}_{P'} \langle (q, v), \sigma \rangle \text{ iff } \langle q_0, \epsilon \rangle \xrightarrow{w'}_{P} \langle q, \sigma \rangle$$

where $h(w) = w'v$. Again this induction proof is left as an exercise. Now, $w \in \mathbf{L}(P')$ iff $\langle q_0', \epsilon \rangle \xrightarrow{w}_{P'} \langle (q, \epsilon), \sigma \rangle$ where $q \in F$ (by definition of PDA acceptance and F') iff $\langle q_0, \epsilon \rangle \xrightarrow{h(w)}_{P} \langle q, \sigma \rangle$ (by exercise) iff $h(w) \in \mathbf{L}(P)$ (by definition of PDA acceptance). Thus, $\mathbf{L}(P') = h^{-1}(\mathbf{L}(P)) = h^{-1}(L)$. □
