1 Pumping Lemma

1.1 Statement and Proof

Pumping Lemma: Overview

Pumping Lemma
Gives the template of an argument that can be used to easily prove that many languages are non-regular.

Pumping Lemma

Lemma 1. *If L is regular then there is a number p (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and*

1. $|y| > 0$
2. $|xy| \leq p$
3. $\forall i \geq 0.\ xy^i z \in L$

Proof. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA such that $L(M) = L$ and let $p = |Q|$. Let $w = w_1 w_2 \cdots w_n \in L$ be such that $n \geq p$. For $1 \leq i \leq n$, let $\{s_i\} = \hat{\delta}_M(q_0, w_1 \cdots w_i)$; define $s_0 = q_0$.

- Since $s_0, s_1, \ldots, s_i, \ldots s_p$ are $p + 1$ states, there must be j, k, $0 \leq j < k \leq p$ such that $s_j = s_k$ ($= q$ say).
- Take $x = w_1 \cdots w_j$, $y = w_{j+1} \cdots w_k$, and $z = w_{k+1} \cdots w_n$
- Observe that since $j < k \leq p$, we have $|xy| \leq p$ and $|y| > 0$.

Claim
For all $i \geq 1$, $\hat{\delta}_M(q_0, xy^i) = \hat{\delta}_M(q_0, x)$.

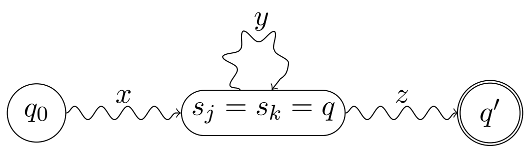
Proof. We will prove it by induction on i.

- *Base Case:* By our assumption that $s_j = s_k$ and the definition of x and y, we have $\hat{\delta}_M(q_0, xy) = \{s_k\} = \{s_j\} = \hat{\delta}_M(q_0, x)$.
- *Induction Step:* We have

$$\begin{aligned}
\hat{\delta}_M(q_0, xy^{\ell+1}) &= \hat{\delta}_M(q, y) \text{ where } \{q\} = \hat{\delta}_M(q_0, xy^\ell) \\
&= \hat{\delta}_M(q, y) \text{ where } \{q\} = \hat{\delta}_M(q_0, x) \\
&= \hat{\delta}_M(q_0, xy) = \hat{\delta}_M(q_0, x)
\end{aligned}$$

□

We now complete the proof of the pumping lemma.



- We have $\hat{\delta}_M(q_0, xy^i) = \hat{\delta}_M(q_0, x)$ for all $i \geq 1$
- Since $w \in L$, we have $\hat{\delta}_M(q_0, w) = \hat{\delta}_M(q_0, xyz) \subseteq F$
- Observe, $\hat{\delta}_M(q_0, xz) = \hat{\delta}_M(q, z) = \hat{\delta}_M(q_0, w)$, where $\{q\} = \hat{\delta}_M(q_0, x) = \hat{\delta}_M(q_0, xy)$. So $xz \in L$
- Similarly, $\hat{\delta}_M(q_0, xy^iz) = \hat{\delta}_M(q_0, xyz) \subseteq F$ and so $xy^iz \in L$ □

Finite Languages and Pumping Lemma

Question
Do finite languages really satisfy the condition in the pumping lemma?

Recall Pumping Lemma: If L is regular then *there is a number* p (the pumping length) such that $\forall w \in L$ with $|w| \geq p$, $\exists x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $|y| > 0$
2. $|xy| \leq p$
3. $\forall i \geq 0.\ xy^iz \in L$

Answer
Yes, they do. Let p be larger than the longest string in the language. Then the condition "$\forall w \in L$ with $|w| \geq p, \ldots$" is *vaccuously* satisfied as there are no strings in the language longer than p!

1.2 Applications

Using the Pumping Lemma
L regular implies that L satisfies the condition in the pumping lemma. If L is not regular *pumping lemma says nothing about L!*

Pumping Lemma, in contrapositive
If L does not satisfy the pumping condition, then L not regular.

Negation of the Pumping Condition

$$\forall p. \quad \exists w \in L.\text{ with } |w| \geq p \quad \forall x, y, z \in \Sigma^*.\, w = xyz$$

$$\left.\begin{array}{ll} (1) & |y| > 0 \\ (2) & |xy| \leq p \\ (3) & \forall i \geq 0.\, xy^i z \in L \end{array}\right\} \text{not all of them hold}$$

Equivalent to showing that if $(1), (2)$ then (3) does not. In other words, we can find i such that $xy^i z \notin L$

Game View

Think of using the Pumping Lemma as a game between you and an *opponent*.

L	Task: To show that L is not regular
$\forall p.$	*Opponent picks* p
$\exists w.$	Pick w that is of length at least p
$\forall x, y, z$	*Opponent divides* w *into* $x, y,$ *and* z *such that* $|y| > 0$, *and* $|xy| \leq p$
$\exists k.$	You pick k and win if $xy^k z \notin L$

Pumping Lemma: If L is regular, *opponent* has a winning strategy (no matter what you do).
Contrapositive: If you can beat the opponent, L not regular.
Your strategy should work for any p and any subdivision that the opponent may come up with.

Example I

Proposition 2. $L_{0n1n} = \{0^n 1^n \mid n \geq 0\}$ *is not regular.*

Proof. Suppose L_{0n1n} is regular. *Let* p *be the pumping length for* L_{0n1n}.

- Consider $w = 0^p 1^p$
- *Since* $|w| > p$, *there are* x, y, z *such that* $w = xyz$, $|xy| \leq p$, $|y| > 0$, *and* $xy^i z \in L_{0n1n}$, *for all* i.
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t 1^p$. Further, as $|y| > 0$, we have $s > 0$.

  $$xy^0 z = 0^r \epsilon 0^t 1^p = 0^{r+t} 1^p$$

  Since $r + t < p$, $xy^0 z \notin L_{0n1n}$. Contradiction! □

Example II

Proposition 3. $L_{\text{eq}} = \{w \in \{0,1\}^* \mid w$ *has an equal number of 0s and 1s*$\}$ *is not regular.*

Proof. Suppose L_{eq} is regular. *Let* p *be the pumping length for* L_{eq}.

- Consider $w = 0^p 1^p$
- *Since* $|w| > p$, *there are* x, y, z *such that* $w = xyz$, $|xy| \leq p$, $|y| > 0$, *and* $xy^i z \in L_{\text{eq}}$, *for all* i.
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t 1^p$. Further, as $|y| > 0$, we have $s > 0$.

$$xy^0 z = 0^r \epsilon 0^t 1^p = 0^{r+t} 1^p$$

  Since $r + t < p$, $xy^0 z \notin L_{\text{eq}}$. Contradiction! □

Example III

Proposition 4. $L_p = \{0^i \mid i \text{ prime}\}$ *is not regular*

Proof. Suppose L_p is regular. *Let* p *be the pumping length for* L_p.

- Consider $w = 0^m$, where $m \geq p + 2$ and m is prime.
- *Since* $|w| > p$, *there are* x, y, z *such that* $w = xyz$, $|xy| \leq p$, $|y| > 0$, *and* $xy^i z \in L_p$, *for all* i.
- Thus, $x = 0^r$, $y = 0^s$ and $z = 0^t$. Further, as $|y| > 0$, we have $s > 0$. $xy^{r+t}z = 0^r(0^s)^{(r+t)}0^t = 0^{r+s(r+t)+t}$. Now $r + s(r + t) + t = (r + t)(s + 1)$. Further $m = r + s + t \geq p + 2$ and $s > 0$ mean that $t \geq 2$ and $s + 1 \geq 2$. Thus, $xy^{r+t}z \notin L_p$. Contradiction! □

Example IV

Question
Is $L_{\text{eq}} = \{xx \mid x \in \{0, 1\}^*\}$ is regular?

Suppose L_{eq} is regular, and let p be the pumping length of L_{eq}.

- Consider $w = 0^p 0^p \in L$.
- Can we find substrings x, y, z satisfying the conditions in the pumping lemma? Yes! Consider $x = \epsilon, y = 00, z = 0^{2p-2}$.
- Does this mean L_{eq} satisfies the pumping lemma? Does it mean it is regular?
  - No! We have chosen a bad w. To prove that the pumping lemma is violated, we only need to exhibit *some* w that cannot be pumped.
- Another bad choice $(01)^p(01)^p$.

Example IV
Reloaded

Proposition 5. $L_{\text{eq}} = \{xx \mid x \in \{0, 1\}^*\}$ *is not regular.*

Proof. Suppose L_{eq} is regular. *Let p be the pumping length for L_{xx}.*

- Consider $w = 0^p10^p1$.
- *Since $|w| > p$, there are x, y, z such that $w = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^iz \in L_p$, for all i.*
- Since $|xy| \leq p$, $x = 0^r$, $y = 0^s$ and $z = 0^t10^p1$. Further, as $|y| > 0$, we have $s > 0$.

$$xy^0z = 0^r\epsilon 0^t10^p1 = 0^{r+t}10^p1$$

Since $r + t < p$, $xy^0z \notin L_{\text{eq}}$. Contradiction! □
