1 Equivalence of Finite Automata and Regular Expressions

Finite Automata Recognize Regular Languages

Theorem 1. *L is a regular language iff there is a regular expression R such that $\mathbf{L}(R) = L$ iff there is a DFA M such that $\mathbf{L}(M) = L$ iff there is a NFA N such that $\mathbf{L}(N) = L$.*

i.e., regular expressions, DFAs and NFAs have the same computational power.

Proof.

- Given regular expression R, will construct *NFA* N such that $\mathbf{L}(N) = \mathbf{L}(R)$
- Given *DFA* M, will construct regular expression R such that $\mathbf{L}(M) = \mathbf{L}(R)$ □

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2 Regular Expressions to NFA

Regular Expressions to Finite Automata

. . . to Non-determinstic Finite Automata

Lemma 2. *For any regex R, there is an NFA N_R s.t. $\mathbf{L}(N_R) = \mathbf{L}(R)$.*

Proof Idea

We will build the NFA N_R for R, inductively, based on the number of operators in R, $\#(R)$.

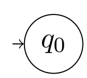
- *Base Case:* $\#(R) = 0$ means that R is $\emptyset, \epsilon$, or a (from some $a \in \Sigma$). We will build NFAs for these cases.
- *Induction Hypothesis:* Assume that for regular expressions R, with $\#(R) < n$, there is an NFA N_R s.t. $\mathbf{L}(N_R) = \mathbf{L}(R)$.
- *Induction Step:* Consider R with $\#(R) = n$. Based on the form of R, the NFA N_R will be built using the induction hypothesis.

Regular Expression to NFA

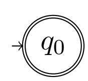
Base Cases

If R is an elementary regular expression, NFA N_R is constructed as follows.

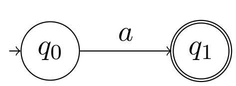
$R = \emptyset$



$R = \epsilon$



$R = a$



Induction Step: Union

Case $R = R_1 \cup R_2$
By induction hypothesis, there are N_1, N_2 s.t. $\mathbf{L}(N_1) = \mathbf{L}(R_1)$ and $\mathbf{L}(N_2) = \mathbf{L}(R_2)$. Build NFA N s.t. $\mathbf{L}(N) = \mathbf{L}(N_1) \cup \mathbf{L}(N_2)$

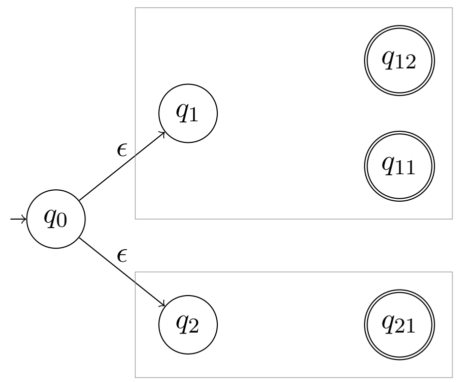


Figure 1: NFA for $\mathbf{L}(N_1) \cup \mathbf{L}(N_2)$

Induction Step: Union
Formal Definition

Case $R = R_1 \cup R_2$
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ (with $Q_1 \cap Q_2 = \emptyset$) be such that $\mathbf{L}(N_1) = \mathbf{L}(R_1)$ and $\mathbf{L}(N_2) = \mathbf{L}(R_2)$. The NFA $N = (Q, \Sigma, \delta, q_0, F)$ is given by

- $Q = Q_1 \cup Q_2 \cup \{q_0\}$, where $q_0 \notin Q_1 \cup Q_2$
- $F = F_1 \cup F_2$
- δ is defined as follows

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1 \\ \delta_2(q, a) & \text{if } q \in Q_2 \\ \{q_1, q_2\} & \text{if } q = q_0 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$

Induction Step: Union
Correctness Proof

Need to show that $w \in \mathbf{L}(N)$ iff $w \in \mathbf{L}(N_1) \cup \mathbf{L}(N_2)$.

$\Rightarrow$ $w \in \mathbf{L}(N)$ implies $q_0 \xrightarrow{w}_N q$ for some $q \in F$. Based on the transitions out of q_0, $q_0 \xrightarrow{\epsilon}_N q_1 \xrightarrow{w}_N q$ or $q_0 \xrightarrow{\epsilon}_N q_2 \xrightarrow{w}_N q$. Consider $q_0 \xrightarrow{\epsilon}_N q_1 \xrightarrow{w}_N q$. (Other case is similar) This means $q_1 \xrightarrow{w}_{N_1} q$ (as N has the same transition as N_1 on the states in Q_1) and $q \in F_1$. This means $w \in \mathbf{L}(N_1)$.

$\Leftarrow$ $w \in \mathbf{L}(N_1) \cup \mathbf{L}(N_2)$. Consider $w \in \mathbf{L}(N_1)$; case of $w \in \mathbf{L}(N_2)$ is similar. Then, $q_1 \xrightarrow{w}_{N_1} q$ for some $q \in F_1$. Thus, $q_0 \xrightarrow{\epsilon}_N q_1 \xrightarrow{w}_N q$, and $q \in F$. This means that $w \in \mathbf{L}(N)$.

Induction Step: Concatenation

Case $R = R_1 \circ R_2$

- By induction hypothesis, there are N_1, N_2 s.t. $\mathbf{L}(N_1) = \mathbf{L}(R_1)$ and $\mathbf{L}(N_2) = \mathbf{L}(R_2)$
- Build NFA N s.t. $\mathbf{L}(N) = \mathbf{L}(N_1) \circ \mathbf{L}(N_2)$

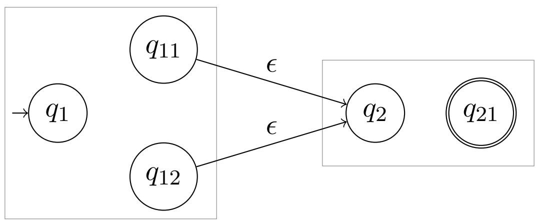


Figure 2: NFA for $\mathbf{L}(N_1) \circ \mathbf{L}(N_2)$

Induction Step: Concatenation
Formal Definition

Case $R = R_1 \circ R_2$
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ (with $Q_1 \cap Q_2 = \emptyset$) be such that $\mathbf{L}(N_1) = \mathbf{L}(R_1)$ and $\mathbf{L}(N_2) = \mathbf{L}(R_2)$. The NFA $N = (Q, \Sigma, \delta, q_0, F)$ is given by

- $Q = Q_1 \cup Q_2$
- $q_0 = q_1$
- $F = F_2$
- δ is defined as follows

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & \text{if } q \in (Q_1 \setminus F_1) \text{ or } a \neq \epsilon \\ \delta_1(q,a) \cup \{q_2\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \delta_2(q,a) & \text{if } q \in Q_2 \\ \emptyset & \text{otherwise} \end{cases}$$

Induction Step: Concatenation
Correctness Proof

Need to show that $w \in \mathbf{L}(N)$ iff $w \in \mathbf{L}(N_1) \circ \mathbf{L}(N_2)$.

$w \in \mathbf{L}(N)$ iff $q_0 \xrightarrow{w}_N q$ for some $q \in F = F_2$. The computation of N on w starts in a state of N_1 (namely, $q_0 = q_1$) and ends in a state of N_2 (namely, $q \in F_2$). The only transitions from a state of N_1 to a state of N_2 is from a state in F_1 which have ϵ-transitions to q_2, the initial state of N_2. Thus, we have

$$q_0 = q_1 \xrightarrow{w}_N q \text{ with } q \in F = F_2$$
$$\text{iff}$$
$$\exists q' \in F_1.\ \exists u, v \in \Sigma^*.\ w = uv \text{ and } q_0 = q_1 \xrightarrow{u}_N q' \xrightarrow{\epsilon}_N q_2 \xrightarrow{v}_N q$$

This means that $q_1 \xrightarrow{u}_{N_1} q'$ (with $q' \in F_1$) and $q_2 \xrightarrow{v}_{N_2} q$ (with $q \in F_2$). Hence, $u \in \mathbf{L}(N_1)$ and $v \in \mathbf{L}(N_2)$, and so $w = uv \in \mathbf{L}(N_1) \circ \mathbf{L}(N_2)$. Conversely, if $u \in \mathbf{L}(N_1)$ and $v \in \mathbf{L}(N_2)$ then for some $q' \in F_1$ and $q \in F_2$, we have $q_1 \xrightarrow{u}_{N_1} q'$ and $q_2 \xrightarrow{v}_{N_2} q$. Then,

$$q_0 = q_1 \xrightarrow{u}_N q' \xrightarrow{\epsilon}_N q_2 \xrightarrow{v}_N q$$

Thus, $q_0 \xrightarrow{w=uv}_N q$ and so $uv \in \mathbf{L}(N)$.

Induction Step: Kleene Closure
First Attempt

Case $R = R_1^*$

- By induction hypothesis, there is N_1 s.t. $\mathbf{L}(N_1) = \mathbf{L}(R_1)$
- Build NFA N s.t. $\mathbf{L}(N) = (\mathbf{L}(N_1))^*$

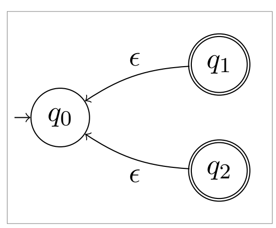


Figure 3: NFA accepts $(\mathbf{L}(N_1))^+$

Problem: May not accept ϵ! One can show that $\mathbf{L}(N) = (\mathbf{L}(N_1))^+$.

Induction Step: Kleene Closure
Second Attempt

Case $R = R_1^*$

- By induction hypothesis, there is N_1 s.t. $\mathbf{L}(N_1) = \mathbf{L}(R_1)$
- Build NFA N s.t. $\mathbf{L}(N) = (\mathbf{L}(N_1))^*$

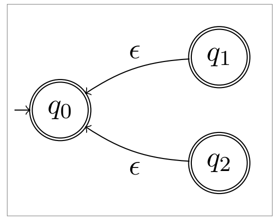


Figure 4: NFA accepts $\supseteq (\mathbf{L}(N_1))^*$

Problem: May accept strings that are not in $(\mathbf{L}(N_1))^*$!

Example demonstrating the problem

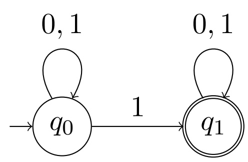


Figure 5: Example NFA N

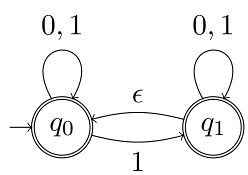


Figure 6: Incorrect Kleene Closure of N

$\mathbf{L}(N) = (0 \cup 1)^*1(0 \cup 1)^*$. Thus, $(\mathbf{L}(N))^* = \epsilon \cup (0 \cup 1)^*1(0 \cup 1)^*$. The previous construction, gives an NFA that accepts $0 \notin (\mathbf{L}(N))^*$!

Induction Step: Kleene Closure
Correct Construction

Case $R = R_1^*$

- First build N_1 s.t. $\mathbf{L}(N_1) = \mathbf{L}(R_1)$
- Given N_1 build NFA N s.t. $\mathbf{L}(N) = \mathbf{L}(N_1)^*$

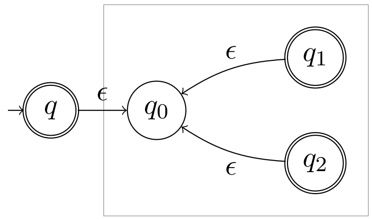


Figure 7: NFA for $\mathbf{L}(N_1)^*$

Induction Step: Kleene Closure
Formal Definition

Case $R = R_1^*$
Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be such that $\mathbf{L}(N_1) = \mathbf{L}(R_1)$. The NFA $N = (Q, \Sigma, \delta, q_0, F)$ is given by

- $Q = Q_1 \cup \{q_0\}$ with $q_0 \notin Q_1$
- $F = F_1 \cup \{q_0\}$
- δ is defined as follows

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & \text{if } q \in (Q_1 \setminus F_1) \text{ or } a \neq \epsilon \\ \delta_1(q,a) \cup \{q_1\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$$

Proof of correctness left as an exercise.

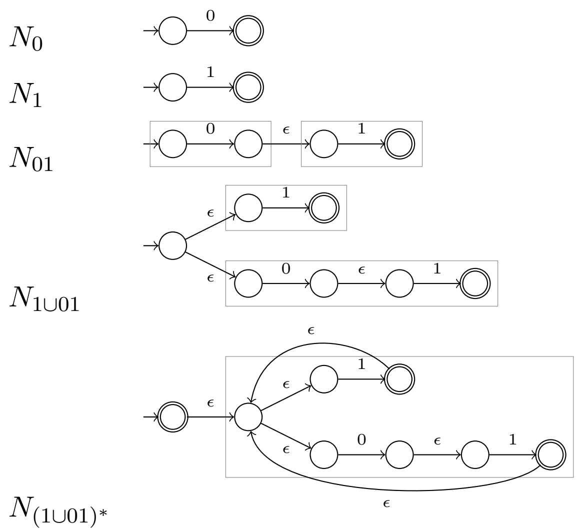
Regular Expressions to NFA
To Summarize

We built an NFA N_R for each regular expression R inductively

- When R was an elementary regular expression, we gave an explicit construction of an NFA recognizing $\mathbf{L}(R)$
- When $R = R_1$ op R_2 (or $R = \text{op}(R_1)$), we constructed an NFA N for R, using the NFAs for R_1 and R_2.

Regular Expressions to NFA
An Example

Build NFA for $(1 \cup 01)^*$



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3 DFAs to Regular Expressions

DFA to Regular Expression

- Given DFA M, will construct regular expression R such that $L(M) = L(R)$. *In two steps:*
  - Construct a "Generalized NFA" (GNFA) G from the DFA M
  - And then convert G to a regex R

3.1 Generalized NFA

Generalized NFA

- A GNFA is similar to an NFA, but:
  - There is a single accept state which is not the start state.
  - The start state has no incoming transitions, and the accept state has no outgoing transitions.
    - These are "cosmetic changes": Any NFA can be converted to an equivalent NFA of this kind.
  - The transitions are labeled not by characters in the alphabet, but by *regular expressions.*
    - For *every* pair of states (q_1, q_2), the transition from q_1 to q_2 is labeled by a regular expression $\rho(q_1, q_2)$.
  - "Generalized NFA" because a normal NFA has transitions labeled by ϵ, elements in Σ (a union of elements, if multiple edges between a pair of states) and $\emptyset$ (missing edges).

Generalized NFA

- Transition: GNFA *non-deterministically* reads a block of characters from the input, chooses an edge from the current state q_1 to another state q_2, and if the block of symbols matches the regex $\rho(q_1, q_2)$, then moves to q_2.
- Acceptance: G accepts w if there exists some sequence of valid transitions such that on starting from the start state, and after finishing the entire input, G is in the accept state.

Generalized NFA: Example

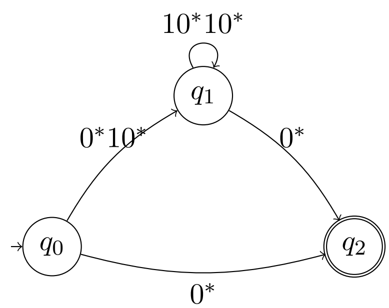

Figure 8: Example GNFA G

Accepting run of G on 11110100 is $q_0 \xrightarrow{1}_G q_1 \xrightarrow{11}_G q_1 \xrightarrow{101}_G q_1 \xrightarrow{00}_G q_2$

Generalized NFA: Definition

Definition 3. A generalized nondeterministic finite automaton (GNFA) is $G = (Q, \Sigma, q_0, q_F, \rho)$, where

- Q is the finite set of states
- Σ is the finite alphabet
- $q_0 \in Q$ initial state
- $q_F \in (Q \setminus \{q_0\}$, a single accepting state
- $\rho : (Q \setminus \{q_F\}) \times (Q \setminus \{q_0\}) \to \mathcal{R}_\Sigma$, where $\mathcal{R}_\Sigma$ is the set of all regular expressions over the alphabet Σ

Generalized NFA: Definition

Definition 4. For a GNFA $M = (Q, \Sigma, q_0, q_F, \rho)$ and string $w \in \Sigma^*$, we say M *accepts* w iff there exist $x_1, \ldots, x_t \in \Sigma^*$ and states $r_0, \ldots, r_t$ such that

- $w = x_1 x_2 x_3 \cdots x_t$
- $r_0 = q_0$ and $r_t = q_F$
- for each $i \in [1, t]$, $x_i \in \mathbf{L}(\rho(r_{i-1}, r_i))$,

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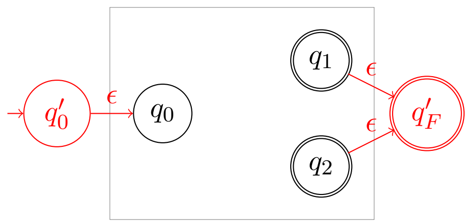
3.2 Converting DFA to GNFA

Converting DFA to GNFA
A DFA $M = (Q, \Sigma, \delta, q_0, F)$ can be easily converted to an equivalent GNFA $G = (Q', \Sigma, q_0', q_F', \rho)$:

- $Q' = Q \cup \{q_0', q_F'\}$ where $Q \cap \{q_0', q_F'\} = \emptyset$
- $\rho(q_1, q_2) = \begin{cases} \epsilon, & \text{if } q_1 = q_0' \text{ and } q_2 = q_0 \\ \epsilon, & \text{if } q_1 \in F \text{ and } q_2 = q_F' \\ \bigcup_{\{a \mid \delta(q_1, a) = q_2\}} a & \text{otherwise} \end{cases}$

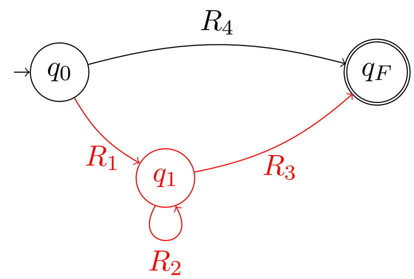


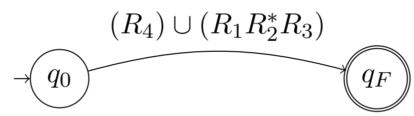
Prove: $\mathbf{L}(G) = \mathbf{L}(M)$.

3.3 Converting GNFA to Regular Expression

GNFA to Regex

- Suppose G is a GNFA with only two states, q_0 and q_F.
- Then $\mathbf{L}(R) = \mathbf{L}(G)$ where $R = \rho(q_0, q_F)$.
- How about G with three states?





- Plan: Reduce any GNFA G with $k > 2$ states to an equivalent GFA with $k - 1$ states.

GNFA to Regex: From k states to $k - 1$ states

Definition 5 (Deleting a GNFA State)**.** Given GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, and any state $q^* \in Q \setminus \{q_0, q_F\}$, define GNFA $\mathsf{rip}(G, q^*) = (Q', \Sigma, q_0, q_F, \rho')$ as follows:

- $Q' = Q \setminus \{q^*\}$.
- For any $(q_1, q_2) \in Q' \setminus \{q_F\} \times Q' \setminus \{q_0\}$ (possibly $q_1 = q_2$), let

$$\rho'(q_1, q_2) = (R_1 R_2^* R_3) \cup R_4,$$

  where $R_1 = \rho(q_1, q^*)$, $R_2 = \rho(q^*, q^*)$, $R_3 = \rho(q^*, q_2)$ and $R_4 = \rho(q_1, q_2)$.

GNFA to Regex: From k states to $k - 1$ states
Correctness

Proposition 6. *For any $q^* \in Q \setminus \{q_0, q_F\}$, G and $\mathsf{rip}(G, q^*)$ are equivalent.*

Proof. Let $G' = \mathsf{rip}(G, q^*)$. We need to show that $\mathbf{L}(G) = \mathbf{L}(G')$. We will prove this in two steps: we will show $\mathbf{L}(G) \subseteq \mathbf{L}(G')$ and then show $\mathbf{L}(G') \subseteq \mathbf{L}(G)$.
$\mathbf{L}(G) \subseteq \mathbf{L}(G')$: First we show $w \in \mathbf{L}(G) \implies w \in \mathbf{L}(G')$. $w \in \mathbf{L}(G) \implies \exists q_0 = r_0, r_1, \ldots, r_t = q_F$ and $x_1, \ldots, x_t \in \Sigma^*$ such that $w = x_1 x_2 x_3 \cdots x_t$ and for each i, $x_i \in L(\rho(r_{i-1}, r_i))$.

We need to show $y_1, \ldots, y_d \in \Sigma^*$ and $q_0 = s_0, s_1, \ldots, s_d = q_F$ such that $w = y_1 \cdots y_d$, and for each i, $y_i \in L(\rho'(s_{i-1}, s_i))$.

Define $(s_0 = q_0, \ldots, s_d = q_F)$ to be the sequence obtained by deleting all occurrences of q^* from $(r_0 = q_0, r_1, \ldots, r_t = q_F)$.

To formally define y_j, first we define σ as follows:

$$\sigma(j) = \begin{cases} 0 & \text{if } j = 0 \\ i & \text{if } 0 < \sigma(j-1) < t, \text{ where } i = \min_{i > \sigma(j-1)} (r_i \neq q^*) \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The range of σ is the set of indices i such that $r_i \neq q^*$. Let $d = \min_k(\sigma(k) = t)$. Then, $s_j = r_{\sigma(j)}$, for $j = 0, \ldots, d$.

Now we define $y_j = x_{\sigma(j-1)+1} \cdots x_{\sigma(j)}$ for $j = 1, \ldots, d$

Then $y_1 \cdots y_d = x_1 \cdots x_t = w$.

We need to show that $y_j \in \mathbf{L}(\rho'(s_{j-1}, s_j))$ for all j. We consider the following cases for j:

- $\sigma(j) = \sigma(j-1) + 1$ (i.e., $r_{\sigma(j-1)+1} \neq q^*$). Then $y_j = x_i$ and $s_{j-1} = r_{i-1}$ and $s_j = r_i$, where $i = \sigma(j)$. $y_j = x_i \in \mathbf{L}(\rho(r_{i-1}, r_i)) \subseteq \mathbf{L}(\rho'(r_{i-1}, r_i)) = \mathbf{L}(\rho'(s_{j-1}, s_j))$.

- $\sigma(j) > \sigma(j-1) + 1$ (i.e., $r_{\sigma(j-1)+1} = q^*$). Then $y_j = x_\ell \cdots x_i$ and $s_{j-1} = r_{\ell-1}$ and $s_j = r_i$, where $\ell = \sigma(j-1) + 1$ and $i = \sigma(j)$.

$$\begin{aligned} y_j = x_\ell \cdots x_i &\in \mathbf{L}(\rho(r_{\ell-1}, r_\ell)\rho(r_\ell, r_{\ell+1}) \cdots \rho(r_{i-1}, r_i)\rho(r_i, r_{i+1})) \\ &= \mathbf{L}(\rho(r_{\ell-1}, q^*)\rho(q^*, q^*)^{i-\ell}\rho(q^*, r_i)) \\ &\subseteq \mathbf{L}(\rho(r_{\ell-1}, r_\ell)\rho(q^*, q^*)^*\rho(q^*, r_i)) \\ &\subseteq \mathbf{L}(\rho(s_{j-1}, q^*)\rho(q^*, q^*)^*\rho(q^*, s_j)) \\ &\subseteq \mathbf{L}(\rho'(s_{j-1}, s_j)) \end{aligned}$$

Thus $w \in \mathbf{L}(G')$ as we set out to prove.

$\mathbf{L}(G') \subseteq \mathbf{L}(G)$: Next we need to show that $w \in \mathbf{L}(G') \implies w \in \mathbf{L}(G)$. $w \in \mathbf{L}(G') \implies \exists q_0 = s_0, s_1, \ldots, s_d = q_F$ and $y_1, \ldots, y_d \in \Sigma^*$ such that $w = y_1 y_2 y_3 \cdots y_d$ and for each j, $y_j \in \mathbf{L}(\rho'(s_{j-1}, s_j)) = \mathbf{L}\left((\rho(s_{j-1}, q^*)\rho(q^*, q^*)^*\rho(q^*, r_i)) \cup \rho(s_{j-1}, s_j)\right)$

Define σ as follows, for $j = 0, \ldots, d$:

$$\sigma(j) = \begin{cases} 0 & \text{if } j = 0, \\ \sigma(j-1) + 1 & \text{if } y_j \in \mathbf{L}(\rho(s_{j-1}, s_j)) \\ \sigma(j-1) + u + 2 & \text{otherwise, where } u = \min_v \left(y_j \in \mathbf{L}(\rho(s_{j-1}, q^*)\rho(q^*, q^*)^v\rho(q^*, s_j))\right) \end{cases}$$

Let $t = \sigma(d)$. For $i = 0, \ldots, t$ define r_i as follows:

$$r(i) = \begin{cases} s_j & \text{if there exists } j \text{ such that} i = \sigma(j), \\ q^* & \text{otherwise.} \end{cases}$$

Finally, define x_i $(i = 1, \ldots, t)$ as follows: if $i = \sigma(j)$ and $i - 1 = \sigma(j-1)$, then let $x_i = y_j$. For other i ($\sigma(j-1) < i - 1 < i \leq \sigma(j)$ for some j), we have $y_j \in \mathbf{L}(\rho(s_{j-1}, q^*)\rho(q^*, q^*)^u\rho(q^*, s_j))$ where $u = \sigma(j) - \sigma(j-1) - 2$. Therefore we can write $y_j = x_\ell \cdots x_{\sigma(j)}$, where $\ell = \sigma(j-1) + 1$, such that $x_\ell \in \mathbf{L}(\rho(s_{j-1}, q^*))$, $x_{\sigma(j)} \in \mathbf{L}(\rho(q^*, s_j))$ and $x_{\ell+1}, \ldots, x_{\sigma(j)-1} \in \mathbf{L}(\rho(q^*, q^*))$. Verify that all x_i $(i = 1, \ldots, t)$ are well-defined by this.

With these definitions it can be easily verified that $x_0 \cdots x_t = y_0 \cdots y_d = w$ and $x_i \in \mathbf{L}(\rho(r_{i-1}, r_i))$.

□

DFA to Regex: Summary

Lemma 7. *For every DFA M, there is a regular expression R such that $\mathbf{L}(M) = \mathbf{L}(R)$.*

- Any DFA can be converted into an equivalent GNFA. So let G be a GNFA s.t. $\mathbf{L}(M) = \mathbf{L}(G)$.
- For any GNFA $G = (Q, \Sigma, q_0, q_F, \rho)$ with $|Q| > 2$, for any $q^* \in Q \setminus \{q_0, q_F\}$, G and $\mathsf{rip}(G, q^*)$ are equivalent. $\mathsf{rip}(G, q^*)$ has one fewer state than G.
- So given G, by applying rip repeatedly (choosing q^* arbitrarily each time), we can get a GNFA G' with two states s.t. $\mathbf{L}(G) = \mathbf{L}(G')$. Formally, by induction on the number of states in G.
- For a 2-state GNFA G', $\mathbf{L}(G') = \mathbf{L}(R)$, where $R = \rho(q_0, q_F)$.

DFA to Regex: Example

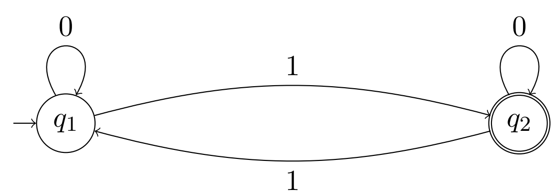


Figure 9: Example DFA D

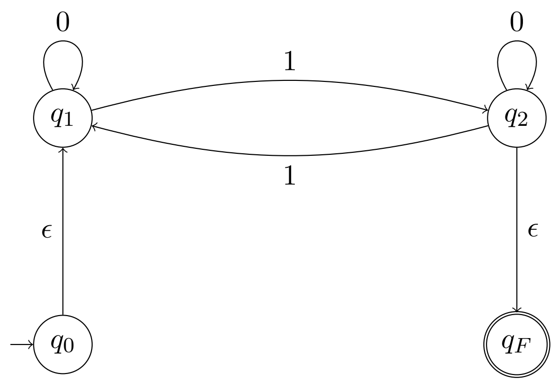


Figure 10: GNFA G equivalent to D, ignoring transitions labelled $\emptyset$

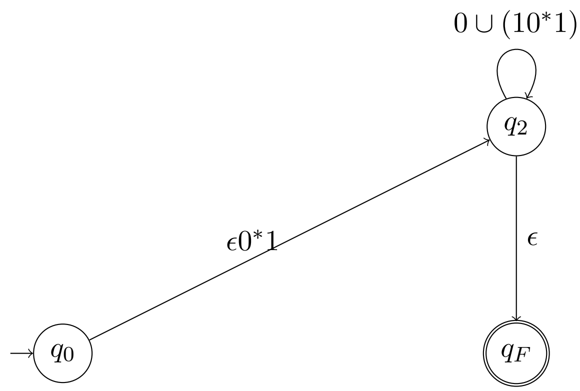


Figure 11: Ripping q_1

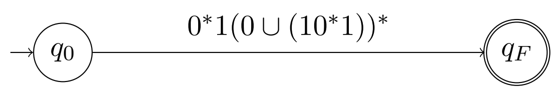


Figure 12: Ripping q_2