Recap

- (Ch 9) Inferring a probability model from a dataset
  - Maximum likelihood estimation (MLE)
  - Confidence intervals for MLE estimates

Today

- (Ch 9) Inferring a probability model from a dataset
  - Bayesian inference
  - Conjugate priors
- Review of eigenvalues, eigenvectors and diagonalization

Maximum likelihood estimation (MLE)

- We write the probability of seeing the data D given parameters θ

$$L(\theta) = P(D|\theta)$$

- The **likelihood function** $L(\theta)$ is not a probability distribution

- The **maximum likelihood estimate** of θ is

$$\hat{\theta} = \arg\max_{\theta} L(\theta)$$

MLE: binomial example

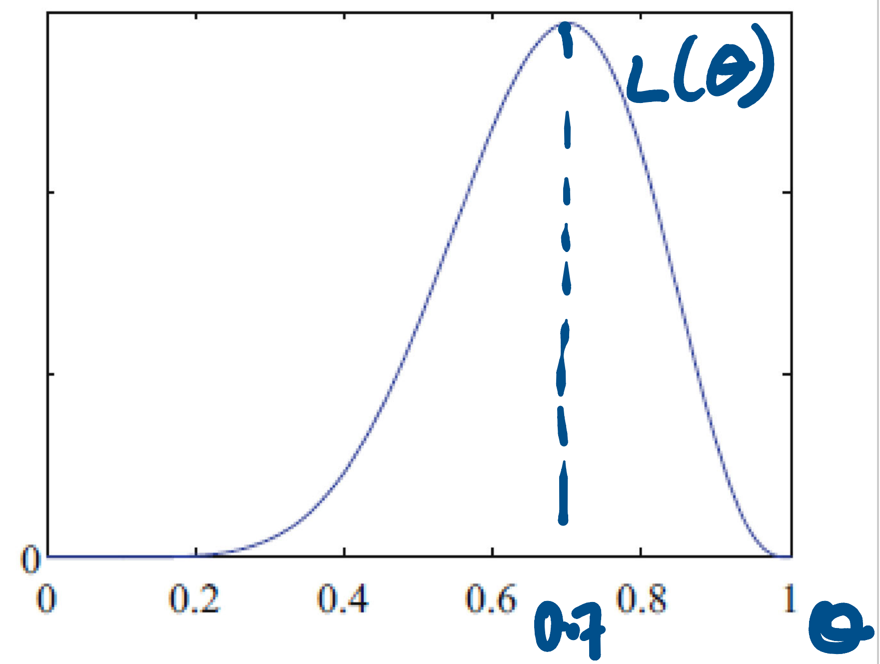
- Suppose we have a coin of unknown probability θ of heads

- We toss it 10 times and observe 7 heads

- The likelihood function is

$$L(\theta) = P(D|\theta) = \binom{10}{7} \theta^7 (1-\theta)^3$$

- The MLE is $\hat{\theta} = 0.7$



Drawbacks of MLE

- Maximizing some likelihood or log-likelihood functions is intractable

- If there isn't much data, the MLE estimate may be unreliable

  - If we observe 3 heads in 10 coin tosses, should we accept that $P(\text{heads}) = 0.3$?

  - If we observe 0 heads in 2 coin tosses, should we accept that $P(\text{heads}) = 0$?

Bayesian inference

- In MLE, we maximized the likelihood function $L(\theta) = P(D|\theta)$
- In Bayesian inference, we will maximize the **posterior**, which is the probability of the parameters θ given the observed data D

$$P(\theta|D)$$

- Unlike $L(\theta)$, the posterior is a probability distribution
- The value of θ that maximizes $P(\theta|D)$ is called the **maximum a posteriori** (MAP) estimate $\hat{\theta}$

The prior

- From Bayes rule

posterior likelihood prior

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)} \propto P(D|\theta)P(\theta)$$

sometimes

- We ignore the probability of the data $P(D)$ because it is constant
- Bayesian inference allows us to incorporate prior beliefs about θ in the **prior** $P(\theta)$, which is useful
  - when we have some beliefs, such as a coin cannot have $P(\text{heads}) = 0$
  - when there isn't much data

Bayesian inference: discrete prior example

- Suppose we have a coin of unknown probability θ of heads
  - We see heads 7 times in 10 tosses as the data D
  - Say we also have prior information about θ: $P(\theta) = \begin{cases} {}^{2}/_{3} & \text{if } \theta = 0.5 \\ {}^{1}/_{3} & \text{if } \theta = 0.6 \\ 0 & \text{otherwise} \end{cases}$

- Applying Bayes rule with $P(D|\theta) = \binom{10}{7}\theta^7(1-\theta)^3$ gives

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

$$P(\theta|D) = \begin{cases} 0.52 & \text{if } \theta = 0.5 \\ 0.48 & \text{if } \theta = 0.6 \\ 0 & \text{otherwise} \end{cases}$$

MAP estimate

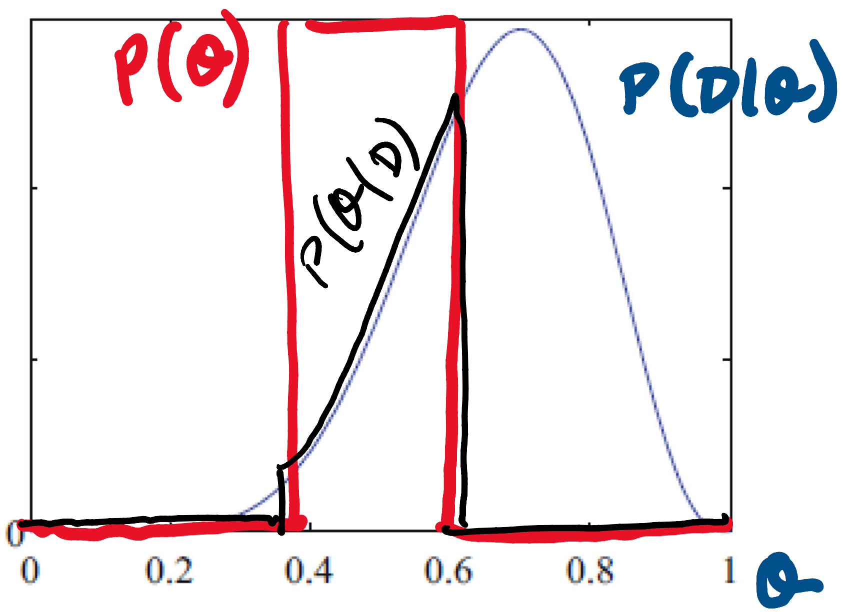
$$\hat{\theta} = 0.5$$

$$P(D) = \sum_{\theta} P(D|\theta)P(\theta)$$

Bayesian inference: continuous prior example

- Suppose we have a coin of unknown probability θ of heads
  - We see heads 7 times in 10 tosses as the data D
  - Say we also have prior information about θ: $P(\theta) = \begin{cases} 5 & \text{if } \theta \in [0.4, 0.6] \\ 0 & \text{if } \theta \notin [0.4, 0.6] \end{cases}$

vertical axis is not to scale



$P(D|\theta) = L(\theta)$

$P(\theta|D) \propto P(D|\theta)P(\theta)$

MAP estimate

$\hat{\theta} = 0.6$

Drawbacks of Bayesian inference

- Maximizing some posteriors $P(\theta|D)$ is intractable
- Some choices of prior $P(\theta)$ can overwhelm any data you observe
- It is hard to justify a choice of prior $P(\theta)$

Conjugate priors

- For a given likelihood function $P(D|\theta)$, a conjugate prior $P(\theta)$ has the following properties
  - The prior $P(\theta)$ belongs to a family of distributions that are expressive
  - The posterior $P(\theta|D) \propto P(D|\theta)P(\theta)$ belongs to the same family as $P(\theta)$
  - The posterior $P(\theta|D)$ is easy to maximize

- We will illustrate these properties for the binomial likelihood, which has a prior called the Beta distribution

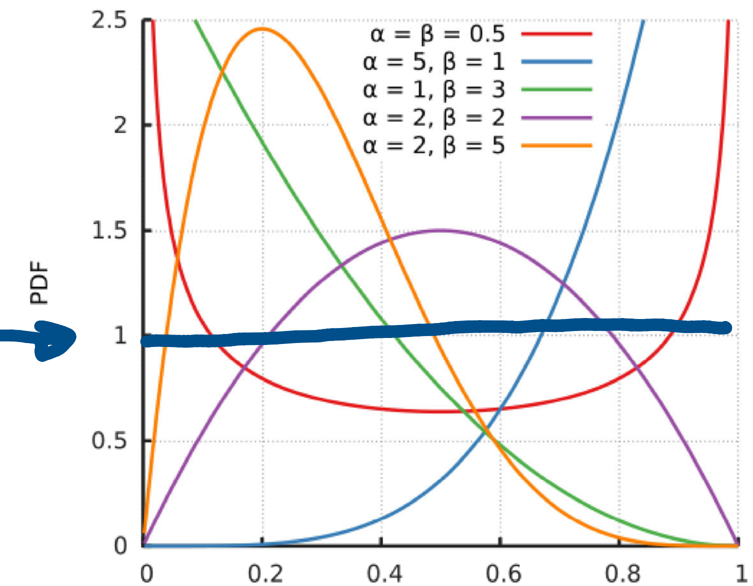
Conjugate prior is expressive $L(\theta) = \binom{N}{k} \theta^k (1-\theta)^{N-k}$

- The conjugate prior for a binomial likelihood is a Beta(α, β) distribution

$$P(\theta) = \mathrm{K}(\alpha,\beta)\theta^{\alpha-1}(1-\theta)^{\beta-1} \quad \text{where } \mathrm{K}(\alpha,\beta) \text{ is a constant}$$

- Beta(α, β) can express a variety of shapes

- Beta$(\alpha = 1, \beta = 1)$ is uniform



Source: Wikipedia

Posterior is in same family as conjugate prior

- The likelihood is Binomial(N, k) and the prior is Beta(α, β)

$$P(D|\theta) = \binom{N}{k} \theta^k (1-\theta)^{N-k}$$

$$P(\theta) = \mathrm{K}(\alpha, \beta) \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

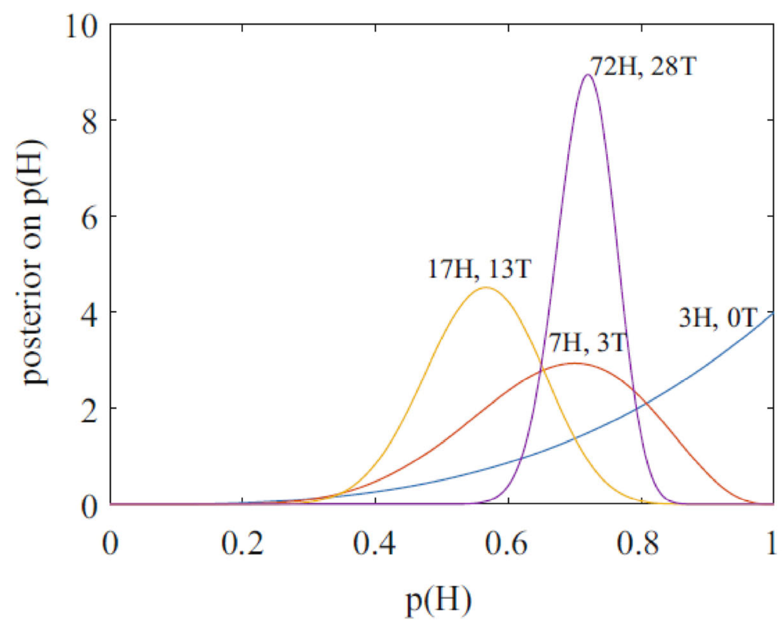
- Then the posterior is Beta$(\alpha + k, \beta + N - k)$

$$P(\theta|D) = \mathrm{K}(\alpha + k, \beta + N - k) \theta^{\alpha+k-1} (1-\theta)^{\beta+N-k-1}$$

Updating the posterior

- Since the posterior is in the same family as the conjugate prior, the posterior can be used as a new prior if more data is observed

- Suppose we start with a uniform prior on the probability θ of heads
  - Then we observe 3H 0T
  - Then we observe 4H 3T for 7H 3T in total
  - Then we observe 10H 10T for 17H 13T in total
  - Then we observe 55H 15T for 72H 28T in total



Posterior is easy to maximize

- The posterior is Beta$(\alpha + k, \beta + N - k)$

$$P(\theta|D) = \mathrm{K}(\alpha + k, \beta + N - k)\theta^{\alpha+k-1}(1-\theta)^{\beta+N-k-1}$$

- Differentiating and setting to 0 gives the MAP estimate

$$\hat{\theta} = \frac{\alpha - 1 + k}{\alpha + \beta - 2 + N}$$

Conjugate priors for other likelihood functions

- If the likelihood is Bernoulli or geometric, the conjugate prior is Beta
- If the likelihood is Poisson or exponential, the conjugate prior is Gamma
- If the likelihood is normal with known variance, the conjugate prior is normal

$Av - \lambda v = 0$

$(A - I\lambda)v = 0$

Eigenvalues and eigenvectors review

- If A is an $n \times n$ square matrix, an eigenvalue λ and its corresponding eigenvector $\boldsymbol{v}$ (of dimension $n \times 1$) have the property that $A\boldsymbol{v} = \lambda\boldsymbol{v}$
- To solve for λ, we solve the characteristic equation $|A - \lambda I| = 0$
- Given a value of λ, we find the corresponding eigenvector(s) by solving $(A - \lambda I)\boldsymbol{v} = 0$
- Note that if $\boldsymbol{v}$ is an eigenvector for λ, then so is any multiple $k\boldsymbol{v}$

Eigenvalues and eigenvectors: example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 3^2 = \lambda^2 - 10\lambda + 25 - 9$$

$$= \lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 8)(\lambda - 2) = 0$$

So eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = 2$

For $\lambda_1 = 8$

$$A - 8I = \begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$

$$A - 2I = \begin{bmatrix} 5-2 & 3 \\ 3 & 5-2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Diagonalization of a symmetric matrix

- If A is an $n \times n$ **symmetric** square matrix, the eigenvalues are real
- If the eigenvalues are also distinct, their eigenvectors are orthogonal
- We can then scale the eigenvectors $\boldsymbol{v}_i$ to ones of unit length $\boldsymbol{u}_i$ and place them into an orthogonal matrix $U = [\boldsymbol{u}_1\ \boldsymbol{u}_2\ \dots\ \boldsymbol{u}_n]$
- Then we can write a diagonal matrix $\Lambda = U^T A U$ such that the diagonal entries of Λ are $\lambda_1, \lambda_2, \dots, \lambda_n$ in that order

Diagonalization example

Diagonalize $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$$\lambda_1 = 8 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow u_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = 2 \Rightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow u_2 = \frac{1}{\|v_2\|} v_2 = \begin{bmatrix} -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}}_{\Lambda} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U^T} \underbrace{\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U}$$