Recap

- (Ch 4) Random variables
  - Weak law of large numbers
  - Simulation examples
  - Continuous random variables

Today

- (Ch 4) Continuous random variables
- (Ch 5) Useful probability distributions
Probability density function (pdf)

• For a continuous random variable $X$, the probability that $X = x$ is essentially zero for all (or most) $x$, so we can’t define $P(X = x)$

• Instead, we define the **probability density function (pdf)** over an infinitesimally small interval $dx$

$$p(x)dx = P(X \in [x, x + dx])$$

• For $a < b$

$$\int_{a}^{b} p(x) \, dx = P(X \in [a, b])$$
Properties of the probability density function

- $p(x)$ is a bit like a discrete random variable’s probability distribution
  - $p(x) \geq 0$ for all $x$
  - The probability of $X$ taking some value is 1

$$\int_{-\infty}^{\infty} p(x) \, dx = 1$$

- $p(x)$ is not like a discrete random variable’s probability distribution
  - $p(x)$ is not the probability that $X = x$
  - $p(x)$ can exceed 1
Probability density function: height example

• Suppose we heard that Napoleon was 62.5 inches tall, rounded up to the nearest half inch. What is the pdf of his height $H$?

• Assume that $H$ is equally likely to be any value in $(62, 62.5]$ inches

$$p(h) = \begin{cases} 
2 & \text{if } h \in (62, 62.5] \\
0 & \text{if } h \notin (62, 62.5] 
\end{cases}$$

where $c$ is a constant

• Then

$$1 = \int_{-\infty}^{\infty} p(h) \, dh = \int_{62}^{62.5} c \, dh = \frac{c}{2} \implies c = 2$$
Probability density function: height example

• What is the probability that $62.1 \leq H \leq 62.2$?

$$P(62.1 \leq H \leq 62.2) = \int_{62.1}^{62.2} p(h) \, dh = \int_{62.1}^{62.2} 2 \, dh = 2h \bigg|_{62.1}^{62.2} = \frac{1}{5}$$

• What is the probability that $H = 62.1$?

$$P(H = 62.1) = \int_{62.1}^{62.1} 2 \, dh = 0$$
Expected value

\[ E[x] = \sum_{x} x p(x) \]

- Expected value of a continuous random variable \( X \)

\[ E[X] = \int_{-\infty}^{\infty} x p(x) \, dx \]

- Expected value of function of continuous random variable \( Y = f(X) \)

\[ E[Y] = E[f(X)] = \int_{-\infty}^{\infty} f(x)p(x) \, dx \]
Expected value: height example

• We have that Napoleon’s height $H$ has probability density function

$$p(h) = \begin{cases} 
2 & \text{if } h \in (62, 62.5] \\
0 & \text{if } h \not\in (62, 62.5] 
\end{cases}$$

• Then the expected value of his height is

$$E[H] = \int_{-\infty}^{\infty} hp(h) \, dh = \int_{62}^{62.5} 2h \, dh = [h^2]_{62}^{62.5} = 62.25$$
Useful probability distributions

• Many common processes generate data with probability distributions that belong to families with known properties

• Even if the data are not distributed according to a known probability distribution, it is sometimes useful in practice to approximate
Discrete uniform distribution

- A discrete random variable $X$ is **uniform** if it takes $k$ different values and
  \[ P(X = x_i) = \frac{1}{k} \quad \text{for all } x_i \text{ that are allowable values} \]

- Examples
  - Rolling a **fair** $k$-sided die
  - Tossing a **fair** coin ($k = 2$)
Bernoulli distribution

• A random variable $X$ is **Bernoulli** if it takes on two values 0 and 1 such that

$$P(X = 1) = p \quad \text{and} \quad P(X = 0) = 1 - p$$

with

$$E[X] = p \quad \text{and} \quad \text{var}[X] = p(1 - p)$$

• Examples
  • Tossing a biased (or fair) coin
  • Making a free throw
  • Rolling a six-sided die and checking if it comes up 6 or not
  • Any indicator function of a random variable
Geometric distribution

- Examples
  - How many rolls of a six-sided die will it take to see the first 6?
  - How many free throws must I attempt to score my first point?
  - How many Bernoulli trials must take place before the first 1 occurs?

- A discrete random variable $X$ is **geometric** if

\[ P(X = k) = (1 - p)^{k-1}p \quad \text{for integer } k \geq 1 \]

with \[ E[X] = \frac{1}{p} \quad \text{and} \quad \text{var}[X] = \frac{1-p}{p^2} \]
Derivation of geometric expected value

\[ E[x] = \sum_{k=1}^{\infty} k (1-\rho)^{k-1} \rho \]

\[ = 1 \rho + 2 (1-\rho) \rho + 3 (1-\rho)^2 \rho + \cdots \]

\[ = (1 + (1-\rho) + (1-\rho)^2 + \cdots) \cdot (1 - (1-\rho) + (1-\rho)^2 + \cdots) \]

\[ = \frac{1}{1-(1-\rho)} \cdot \frac{1}{1-(1-\rho)} \]

\[ E[x] = 1 + (1-\rho) E[x] \Rightarrow E[x] = \frac{1}{\rho} \]
Geometric distribution: die example

- Let $X$ be the number of rolls of a fair six-sided die needed to see the first 6. What is $P(X = k)$ for $k = 1, 2, 3$?

  $P(X=1) = (1-p)p = \frac{1}{6} \approx 0.167$

  $P(X=2) = (1-p)^{1} p = \frac{5}{6} \cdot \frac{1}{6} \approx 0.139$

  $P(X=3) = (1-p)^{2} p = \left(\frac{5}{6}\right)^{2} \cdot \frac{1}{6} \approx 0.116$

- Calculate $E[X]$ and $\text{std}[X]$

  $E[X] = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$

  $\text{std}[X] = \sqrt{\frac{1-p}{p^2}} = \sqrt{\frac{\frac{5}{6}}{\left(\frac{1}{6}\right)^2}} = \sqrt{30} \approx 5.48$
Binomial distribution

- Examples
  - If we roll a six-sided die $N$ times, how many sixes will we see?
  - If I attempt $N$ free throws, how many points will I score?
  - What is the sum of $N$ independent and identically distributed Bernoulli trials?

- A discrete random variable $X$ is binomial if

\[
P(X = k) = \binom{N}{k} p^k (1 - p)^{N-k} \quad \text{for integer } 0 \leq k \leq N
\]

with $E[X] = Np$ and $\text{var}[X] = Np(1 - p)$
Binomial distribution: die example

• Let $X$ be the number of sixes in 36 rolls of a fair six-sided die. What is $P(X = k)$ for $k = 5, 6, 7$?

$$P(X = 5) = \binom{36}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^{31} \approx 0.170$$  
$$P(X = 7) = \binom{36}{7} \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^{29} \approx 0.151$$

$$P(X = 6) = \binom{36}{6} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{30} \approx 0.176$$

• Calculate $E[X]$ and $\text{std}[X]$

$$E[X] = (36)\left(\frac{1}{6}\right) = 6$$  
$$\text{std}[X] = \sqrt{Np(1-p)} = \sqrt{36 \cdot \frac{1}{6} \cdot \frac{5}{6}} = \sqrt{5} \approx 2.24$$
Betting brainteaser

• What would you rather bet on?
  • How many rolls of a fair six-sided die will it take to see the first 6?
    ✔ How many sixes will appear in 36 rolls of a fair six-sided die?

• Why?

Geometric: \( P(1) \approx 0.167 \)

Binomial: \( P(6) \approx 0.176 \) ✔
Multinomial distribution

• Examples
  • If we roll a six-sided die $N$ times, how many of each value will we see?
  • What are the counts of $N$ independent and identically distributed trials?

• A discrete $k$-tuple random variable $(X_1, X_2, ..., X_k)$ is **multinomial** if

\[
P(X_1 = n_1, X_2 = n_2, ..., X_k = n_k) = \frac{N!}{n_1!n_2!...n_k!} p_1^{n_1} p_2^{n_2} ... p_k^{n_k}
\]

where $N = n_1 + n_2 + \cdots + n_k$
Multinomial distribution: die example

What is the probability of seeing 1 one, 2 twos, 3 threes, 4 fours, 5 fives and 0 sixes in 15 rolls of a fair six-sided die?

\[
p(x_1=1, x_2=2, x_3=3, x_4=4, x_5=5, x_6=0)
\]

\[
= \frac{15!}{1!2!3!4!5!0!} \left( \frac{1}{6} \right)^1 \left( \frac{1}{6} \right)^2 \left( \frac{1}{6} \right)^3 \left( \frac{1}{6} \right)^4 \left( \frac{1}{6} \right)^5 \left( \frac{1}{6} \right)^0
\]

\[
= \frac{15!}{2!3!4!5!} \left( \frac{1}{6} \right)^5
\]