Recap

• (Ch 3) Random outcomes and events

Today

• (Ch 4) Random variables
Conditional probability and independence

• The definition of conditional probability is

\[ P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \]

• If \( E_1 \) and \( E_2 \) are independent, then \( P(E_1 \cap E_2) = P(E_1)P(E_2) \), so

\[ P(E_2|E_1) = P(E_2) \]
Random variables

• A random variable is a function that maps events to real numbers.

• Example: Toss a coin. Let random variable $X$ be
  • 0 if the coin comes up heads
  • 1 if the coin comes up tails

  \[
  \text{Let } X = \begin{cases} 
  0 & \text{with probability 0.5} \\
  1 & \text{with probability 0.5}
  \end{cases}
  \]

• Random variables have nothing to do with variables in a program, even though we might use a programming variable to represent a random variable.
Random variables: more examples

• Number of pairs in a hand of 5 cards
  • Let a single outcome be the hand of cards
  • Each outcome maps to a number from 0 to 2

• Number of electoral votes that a US presidential candidate will win
  • Let a single outcome be the list of votes or non-votes of all registered voters
  • Each outcome maps to a number from 0 to 538
Random variables and events

- Let $X$ be a random variable

- The set of outcomes $\{A \mid X(A) = x\}$ is an event with probability $P(X = x)$

- Likewise, the set of outcomes $\{A \mid X(A) \leq x\}$ is an event with probability $P(X \leq x)$
Random variables and events: dice example

• Roll 2 three-sided dice

• How many outcomes?

• Define the following random variables
  • Let $X$ be the value of die 1
  • Let $Y$ be the value of die 2
  • Let sum $S = X + Y$
  • Let difference $D = X - Y$
Random variables and events: dice example

- Calculate the following probabilities

- \( P(X = 1) = \frac{1}{3} \)
- \( P(Y \leq 2) = \frac{2}{3} \)
- \( P(S = 5) = \frac{2}{9} \)
- \( P(D \leq -1) = \frac{8}{9} = \frac{1}{2} \)
Probability distribution

- $P(X = x)$ is called the **probability distribution** of $X$

- $P(X = x)$ is also denoted as $P(x)$ or $p(x)$

- $P(X = x) \geq 0$ for all values that $X$ can take and is 0 everywhere else

- The sum of the probability distribution $\sum_x P(x) = 1$ because
  - $\{A \mid X(A) = x_i\}$ and $\{A \mid X(A) = x_j\}$ are disjoint for $x_i \neq x_j$
  - $\{A \mid X(A) = x_i\}$ cover the sample space $\Omega$
Cumulative distribution function

- $P(X \leq x)$ is called the **cumulative distribution function** of $X$

- $P(X \leq x)$ is also denoted as $f(x)$

- $P(X \leq x)$ is a non-decreasing function of $x$
Distribution functions: dice example

\[ P(S = s) = p(s) \]

\[ P(S \leq s) = f(s) \]
Joint probability distribution

• The joint probability distribution of two random variables \( X \) and \( Y \) is \( P(\{X = x\} \cap \{Y = y\}) \), also denoted \( P(x, y) \) for short.

• We can recover the individual probability distributions \( P(x) \) and \( P(y) \) from the joint probability distribution as follows:

\[
P(x) = \sum_y P(x, y) \quad \text{and} \quad P(y) = \sum_x P(x, y)
\]

• The sum of the joint probability distribution \( \sum_y \sum_x P(x, y) = 1 \).
Joint probability distribution: dice example

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<th>Y = 1</th>
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$p(x)$ and $p(y)$ values:

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Joint probability distribution: dice example

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- \( P(s,d) \)
- \( P(s) \)

\( P(d) \):
- 1/9
- 2/9
- 3/9
- 4/9
- 5/9

\( P(s) \):
- 1/4
- 1/4
- 3/4
- 3/4
- 1/4
- 1/4

1. 1/9
2. 2/9
3. 3/9
4. 4/9
5. 5/9
6. 6/9
Independence of random variables

- Random variables $X$ and $Y$ are independent if

$$P(x, y) = P(x)P(y) \text{ for all } x \text{ and } y$$

- For the dice example, are the following variables independent?

  - $X$ and $Y$ **Yes**
  - $S$ and $D$ **No** a.g. $P(s=2, d=-2) \neq P(s=2)P(d=-2)$
Conditional probability distribution

- The **conditional probability distribution** of $X$ given $Y$ is

\[ P(x|y) = \frac{P(x, y)}{P(y)} \]

- For any given $y$, \( \sum_x P(x|y) = 1 \)

- If $X$ and $Y$ are independent, $P(x, y) = P(x)P(y)$, so $P(x|y) = P(x)$
Conditional distribution: dice example

\[
P(s|d) = \frac{P(s,d)}{P(d)}
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<th>P(s)</th>
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Bayes rule for random variables

• Bayes rule for events generalizes to random variables

\[ P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)} \]

• Let’s check Bayes rule for a case of the dice example

\[ P(D = 0|S = 2) = \frac{\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{P(S = 2)} = 1 \]