Recap

- (Ch 3) More on probability
  - How to count without counting
  - Independence

Today

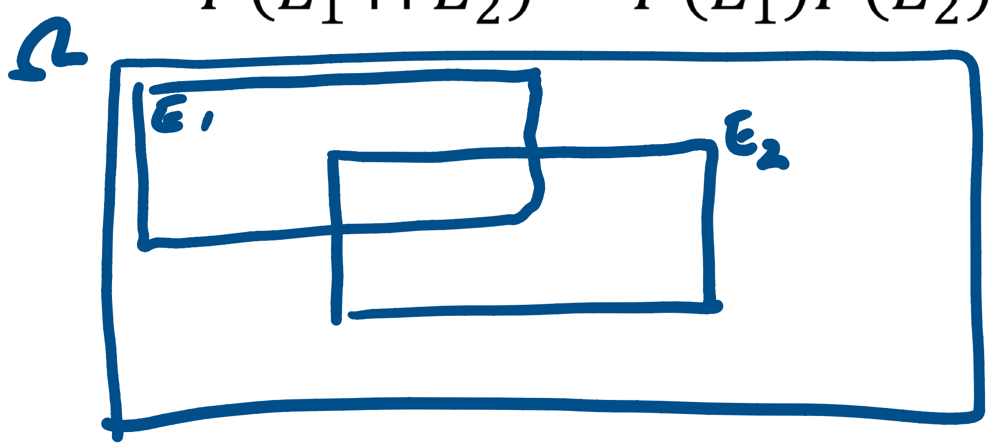
- (Ch 3) How are events related?
  - Independence
  - Conditional probability
  - Conditional independence

Review of independence

- Two events are independent if knowing whether one event happened does not change the probability of the other event

- Definition: Events E_1 and E_2 are **independent** if and only if

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$



Overbooking example 1

An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability p what is the probability that the flight is overbooked?

$P(\{\text{all 7 show up}\}) = p \cdot p \cdots p = p^7$

Overbooking example 2

An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability p what is the probability that exactly 6 passengers show up?

$$P(\{\text{exactly 6 show up}\}) = \binom{8}{6} \underbrace{p^6 (1-p)^2}_{\substack{\text{probability that a} \\ \text{specific 6 show up} \\ \text{and the others don't} \\ \text{show up}}}$$

no. of ways of choosing the specific 6

Overbooking example 3

An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability p what is the probability that the flight is overbooked?

$$P(\{\text{overbooked}\}) = \binom{8}{7} p^7 (1-p) + \binom{8}{8} p^8$$

$$= 8p^7(1-p) + p^8$$

Overbooking example 4

$t > s$

An airline has a flight with s seats. They always sell t tickets for this flight. If ticket holders show up independently with probability p what is the probability that exactly u passengers show up?

$$P(\{\text{exactly } u \text{ show up}\}) = \binom{t}{u} p^u (1-p)^{t-u}$$

Overbooking example 5

An airline has a flight with s seats. They always sell t tickets for this flight. If ticket holders show up independently with probability p what is the probability that the flight is overbooked?

$$P(\{\text{overbooked}\}) = \sum_{u=s+1}^{t} \binom{t}{u} p^u (1-p)^{t-u}$$

Pairwise independence is not independence

- Draw three cards from standard deck with replacement
  - Let E_1 be the event that card 1 and card 2 have the same suit
  - Let E_2 be the event that card 2 and card 3 have the same suit
  - Let E_3 be the event that card 3 and card 1 have the same suit

$P(E_1) = 1/4$

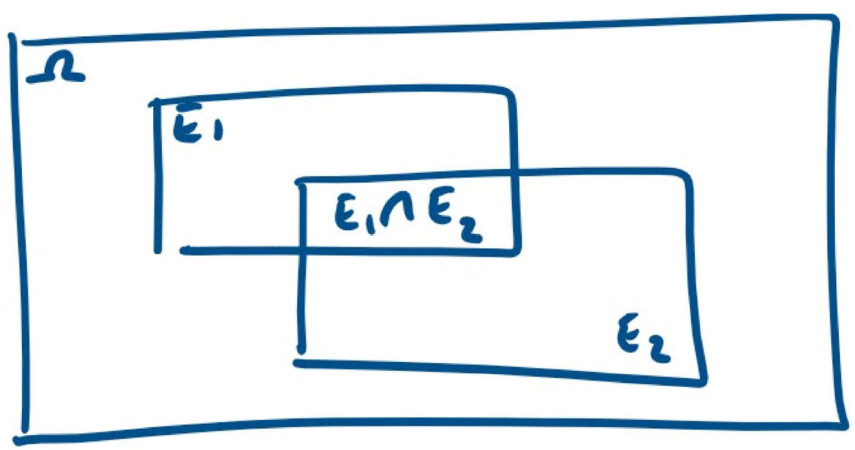
- Are the following events independent?
  - E_1 and E_2?
  - E_2 and E_3?
  - E_3 and E_1?
  - E_1, E_2 and E_3?

E_1, E_2, E_3 are pairwise independent

No, if E_1 & E_2 happened, then E_3 definitely happened

Conditional probability

The **conditional probability** of E_2 given E_1
is the probability of E_2 given that E_1 has happened.

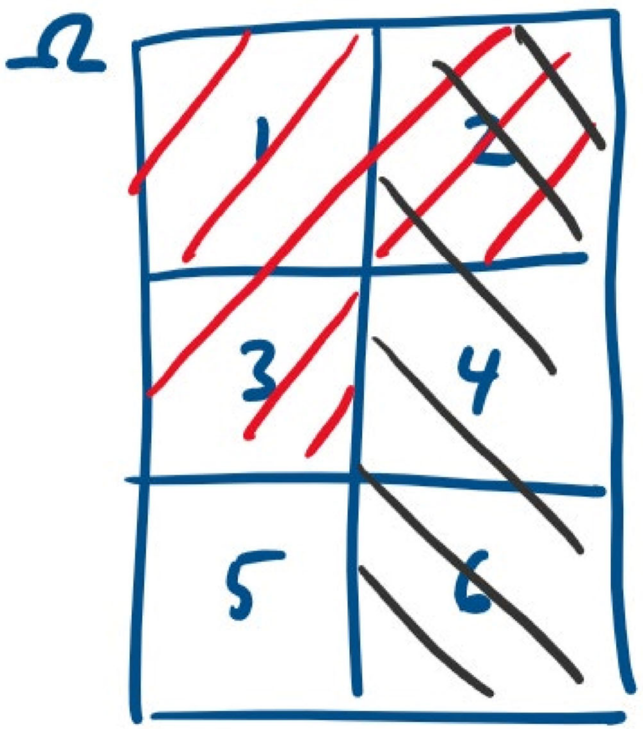


$$P(E_2 | E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

Conditional probability: die example

Suppose you roll a six-sided die

- Let E_1 be the event that it comes up **either 1, 2 or 3**
- Let E_2 be the event that it comes up **even**



$$P(E_2 | E_1) = \frac{1}{3} \neq P(E_2) = \frac{1}{2}$$

Bayes rule for events (simple form)

- The definition of conditional probability implies that

$$P(E_2|E_1)P(E_1) = \underbrace{P(E_1 \cap E_2) = P(E_1|E_2)P(E_2)}_{\text{symmetry}}$$

- Bayes rule

$$P(E_2|E_1) = \frac{P(E_1|E_2)P(E_2)}{P(E_1)}$$

Bayes rule: car example

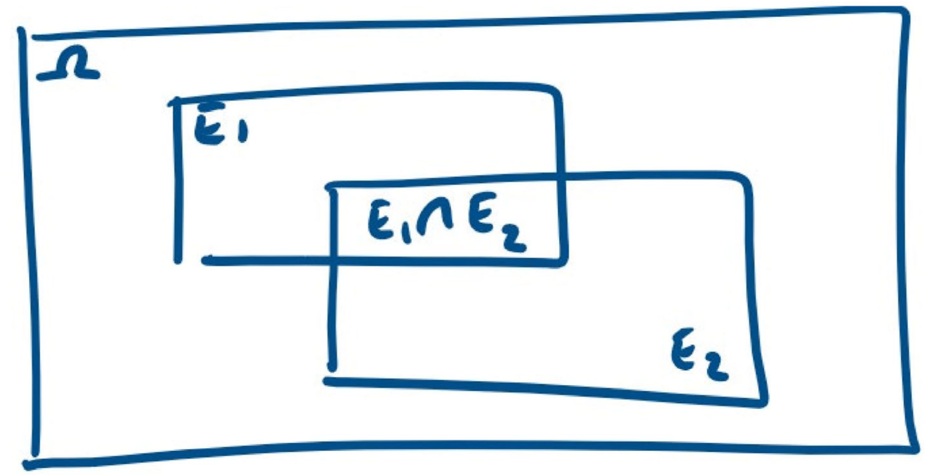
There are two car factories, A and B, that supply the same dealer

- Factory A produced 1000 cars, of which 10 were lemons
- Factory B produced 2 cars and both were lemons

You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

$$P(B|L) = \frac{P(L|B)\,P(B)}{P(L)} = \frac{1 \cdot \frac{2}{1002}}{\frac{12}{1002}} = \frac{1}{6}$$

Total probability



$$P(E_1) = P(E_1 \cap E_2) + P(E_1 \cap E_2^c)$$

$$= P(E_1 | E_2)\,P(E_2) + P(E_1 | E_2^c)\,P(E_2^c)$$

Bayes rule for events (alternative form)

$$P(E_2|E_1) = \frac{P(E_1|E_2)P(E_2)}{P(E_1)} = \overbrace{\frac{P(E_1|E_2)P(E_2)}{P(E_1|E_2)P(E_2) + P\left(E_1\middle|E_2^C\right)P(E_2^C)}}$$

total probability

Bayes rule: false positive example

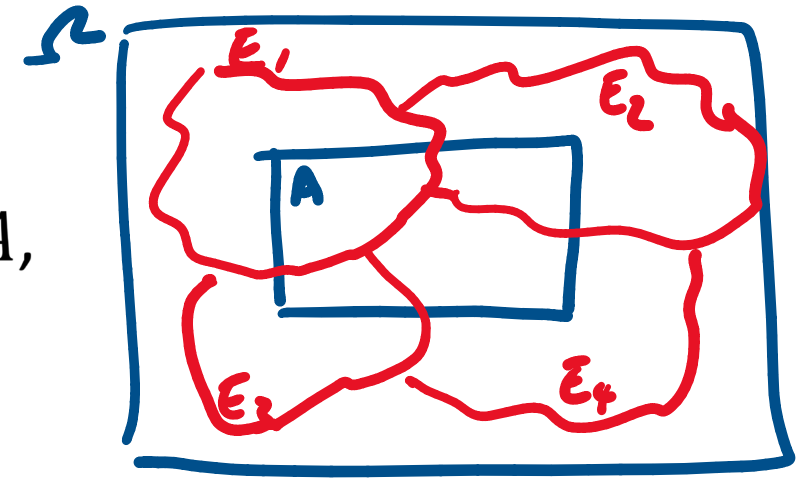
Suppose there is a blood test for a rare disease.

- The disease occurs in 1 in every 100000 people
- If you have the disease, the test will say so with probability 0.95
- If you do not have it, the test will give a false positive with probability 0.001

What is $P(D|T)$, the probability that you have the disease given that you have tested positive?

$$P(D|T) = \frac{\overset{0.95}{P(T|D)}\,\overset{10^{-5}}{P(D)}}{\underset{0.95}{P(T|D)}\,\underset{10^{-5}}{P(D)} + \underset{10^{-3}}{P(T|D^c)}\,\underset{(1-10^{-5})}{P(D^c)}} \approx 0.0094 < 1\,\%$$

Total probability (extended form)



If a set of disjoint events E_j "cover" an event A, then the total probability formula is:

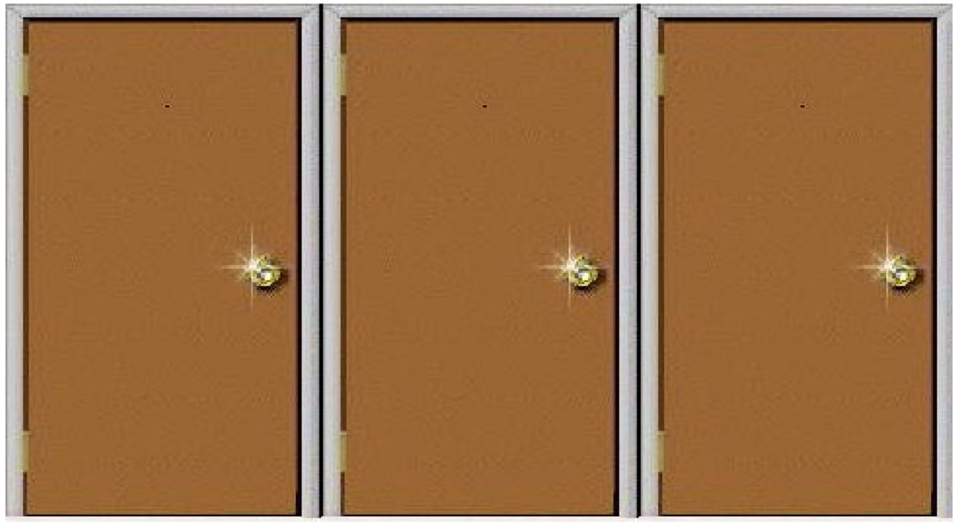
$$P(A) = \sum_j P(A|E_j)P(E_j)$$

Bayes rule for events (extended form)

If a set of disjoint events E_i "cover" an event A,
then Bayes rule is:

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{\sum_j P(A|E_j)P(E_j)}$$

Bayes rule: Monty Hall problem



Demo:

http://onlinestatbook.com/2/probability/monty_hall_demo.html

Bayes rule: Monty Hall problem

- Suppose you first select Door 1 and then Monty opens Door 2 to reveal a goat. Should you stay with Door 1 or switch to Door 3? (Note that this situation is equivalent to any other choices of doors)

- Define the following events
  - Let C_i be the event that the car is behind Door i
  - Let M_i be the event that Monty open Door i to reveal a goat

- We will compare
  - Probability $P(C_1|M_2)$ of winning if you stay with Door 1
  - Probability $P(C_3|M_2)$ of winning if you switch to Door 3

Bayes rule: Monty Hall problem

- Stay with Door 1

$$P(C_1|M_2) = \frac{P(M_2|C_1)\overset{1/3}{P(C_1)}}{\underset{1/2}{P(M_2|C_1)}\underset{1/3}{P(C_1)} + \underset{0}{P(M_2|C_2)}\underset{1/3}{P(C_2)} + \underset{1}{P(M_2|C_3)}\underset{1/3}{P(C_3)}} = \frac{1}{3}$$

- Switch to Door 3

$$P(C_3|M_2) = \frac{\overset{1}{P(M_2|C_3)}\overset{1/3}{P(C_3)}}{\underset{1/2}{P(M_2|C_1)}\underset{1/3}{P(C_1)} + \underset{0}{P(M_2|C_2)}\underset{1/3}{P(C_2)} + \underset{1}{P(M_2|C_3)}\underset{1/3}{P(C_3)}} = \frac{2}{3}$$

Conditional independence

Events E_1 and E_2 are **conditionally independent** given event A if

$$P(E_1 \cap E_2 | A) = P(E_1|A)P(E_2|A) \quad \checkmark$$