Probability

• Probability gives us techniques to reason about uncertain situations
• How likely is it that something will happen?

Today

• (Ch 3) Basic ideas in probability
  • Outcomes, sample space and events
  • Probability axioms and properties
  • Using counting to determine probability
Outcomes and the sample space

• An outcome is a possible result of a random experiment

• The set of all possible outcomes is the sample space $\Omega$
Sample space: coin examples

• Tossing a fair coin?

• Tossing a biased coin?

• Tossing a nickel and a dime?

• Tossing two identical coins?

• Tossing a coin until it comes up heads?
Sample space: sock example

Drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock with replacement?
Sample space: another sock example

Drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock without replacement?
Sample space: real life examples

• Grade in CS 361?

• Cause of death?
Frequency interpretation of probability

• Given an experiment with an outcome called $A$, we can calculate the probability of $A$ by repeating the experiment over and over forever

$$P(A) = \lim_{N \to \infty} \frac{\text{number of time } A \text{ occurs}}{N}$$

• Consequences

$$0 \leq P(A) \leq 1$$

$$\sum_{A_i \in \Omega} P(A_i) = 1$$
Events

• An event $E$ is a subset of the sample space $\Omega$

• That is an event $E$ is a set of outcomes. It may contain:
  • zero outcomes
  • one outcome
  • many outcomes
  • all outcomes
Events: coin examples

• When two coins are tossed:
  • Both coins come up the same?

  • At least one head comes up?

  • At least three heads come up?

• When tossing a coin until it comes up heads:
  • Coin is tossed at least 3 times?
Events: sock examples

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock without replacement:

• Get a matching pair?

• Get a blue sock first or get an orange sock second?
Combining events

Say we will roll a six-sided die. Let $E_1 = \{1, 2, 5\}$ and $E_2 = \{2, 4, 6\}$.

- What is $E_1 \cup E_2$?
- What is $E_1 \cap E_2$?
- What is $E_1 - E_2$?
- What is $E_1^c = \Omega - E_1$?
Axiomatic definition of probability

A probability function is any function $P$ that maps sets to real numbers and satisfies the following axioms

• Probabilities of events are non-negative: $P(E) \geq 0$

• Every experiment has an outcome: $P(\Omega) = 1$

• The probability of disjoint events is additive:

$$P(E_1 \cup E_2 \cup \cdots \cup E_N) = \sum_{i=1}^{N} P(E_i) \text{ if } E_i \cap E_j = \emptyset \text{ for all } i \neq j$$
Using counting to determine probability

• From the last axiom, the probability of event $E$ is the sum of the probabilities of the outcomes $A_i$ it contains:

$$P(E) = \sum_{A_i \in E} P(A_i)$$

• If we have set up our sample space so that it makes sense to assign equal probability to all outcomes $A_i$, then we can use counting to determine probability

$$P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes in } \Omega}$$
Probability: coin examples

• When two coins are tossed:
  • Probability of both coins come up the same?
  • Probability of at least one head comes up?
  • Probability of at least three heads come up?

• When tossing a coin until it comes up heads:
  • Probability of coin being tossed at least 3 times?
Probability: sock examples

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock without replacement:

• Probability of getting a matching pair?

• Probability of getting a blue sock first or an orange sock second?
Properties of probability

\[ P(E^c) = 1 - P(E) \]

\[ P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2) \]
More properties of probability

• \( P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \)

• \( P(E_1 \cup E_2 \cup E_3) \)
  
  \[ = P(E_1) + P(E_2) + P(E_3) \]
  
  \[ - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) \]
  
  \[ + P(E_1 \cap E_2 \cap E_3) \]
Complement property: sock example

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock without replacement:

• Probability of getting a blue sock first or an orange sock second?
Union property: sock example

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock \textbf{without replacement}:

- Probability of getting a blue sock first or an orange sock second?