Probability

- Probability gives us techniques to reason about uncertain situations
- How likely is it that it that something will happen?

Today

- (Ch 3) Basic ideas in probability
  - Outcomes, sample space and events
  - Probability axioms and properties
  - Using counting to determine probability

Outcomes and the sample space

- An outcome is a possible result of a random experiment

- The set of all possible outcomes is the sample space Ω

Sample space: coin examples

- Tossing a fair coin? $\Omega = \{H, T\}$
- Tossing a biased coin?
- Tossing a nickel and a dime? $\Omega = \{HH, HT, TH, TT\}$
- Tossing two identical coins? $\{HH, HT, TT\}$ or $\{HH, HT, TH, TT\}$
- Tossing a coin until it comes up heads?

$\{H, TH, TTH, TTTH, TTTTH, \ldots\}$ (infinite)

Sample space: sock example

Drawing 2 socks one-at-a-time $\{BB, BO, OB, OW, WO, BW, WB, OO, WW\}$
from a bag containing 2 blue socks, 1 orange sock and 1 white sock
with replacement?

or $\{B_1B_1, B_1B_2, B_1O, B_1W,$
$B_2B_1, B_2B_2, B_2O, B_2W,$
$OB_1, OB_2, OO, OW,$
$WB_1, WB_2, WO, WW\}$

Sample space: another sock example

Drawing 2 socks one-at-a-time $\{BB, BO, BW, OB, OW, WB, WO\}$
from a bag containing 2 blue socks, 1 orange sock and 1 white sock
without replacement?

or $\{ B_1B_2, B_1O, B_1W,$

$B_2B_1, B_2O, B_2W,$

$OB_1, OB_2, OW,$

$WB_1, WB_2, WO \}$

Sample space: real life examples

- Grade in CS 361?

- Cause of death?

ICD-11

Frequency interpretation of probability

- Given an experiment with an outcome called A, we can calculate the probability of A by repeating the experiment over and over forever

$$P(A) = \lim_{N\to\infty} \frac{number\ of\ time\ A\ occurs}{N}$$

- Consequences

$$0 \le P(A) \le 1$$

$$\sum_{A_i \in \Omega} P(A_i) = 1$$

Events

- An event E is a subset of the sample space Ω

- That is an event E is a set of outcomes. It may contain:
  - zero outcomes $E = \emptyset$
  - one outcome e.g. $E = \{A_1\}$
  - many outcomes e.g. $E = \{A_1, A_2\}$
  - all outcomes $E = \Omega$

Events: coin examples

- When two coins are tossed:
  - Both coins come up the same? $E = \{HH, TT\}$
  - At least one head comes up? $E = \{HH, HT, TH\}$
  - At least three heads come up? $E = \emptyset$
- When tossing a coin until it comes up heads:
  - Coin is tossed at least 3 times?

$E = \{TTH, TTTH, \ldots\}$ (infinite)

Events: sock examples

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock **without replacement**:

- Get a matching pair?

$E = \{B_1B_2, B_2B_1\}$

- Get a blue sock first or get an orange sock second?

$E = \{B_1B_2, B_1O, B_1W, B_2B_1, B_2O, B_2W, WO\}$

Combining events

$\Omega = \{1,2,3,4,5,6\}$

Say we will roll a six-sided die. Let $E_1 = \{1, 2, 5\}$ and $E_2 = \{2, 4, 6\}$.

- What is $E_1 \cup E_2$? $\{1,2,4,5,6\}$
- What is $E_1 \cap E_2$? $\{2\}$
- What is $E_1 - E_2$? $\{1,5\}$
- What is $E_1^c = \Omega - E_1$? $\{3,4,6\}$

Axiomatic definition of probability

A probability function is any function P that maps sets to real numbers and satisfies the following axioms

- Probabilities of events are non-negative: $P(E) \geq 0$

- Every experiment has an outcome: $P(\Omega) = 1$

- The probability of disjoint events is additive:

$$P(E_1 \cup E_2 \cup \cdots \cup E_N) = \sum_{i=1}^{N} P(E_i) \quad \text{if } E_i \cap E_j = \emptyset \text{ for all } i \neq j$$

Using counting to determine probability

- From the last axiom, the probability of event E is the sum of the probabilities of the outcomes A_i it contains:

$$P(E) = \sum_{A_i \in E} P(A_i)$$

- If we have set up our sample space to that it makes sense to assign equal probability to all outcomes A_i, then we can using counting to determine probability

$$P(E) = \frac{number\ of\ outcomes\ in\ E}{total\ number\ of\ outcomes\ in\ \Omega}$$

Probability: coin examples

- When two coins are tossed:
  - Probability of both coins come up the same? $P(E) = \frac{2}{4} = \frac{1}{2}$
  - Probability of at least one head comes up? $P(E) = \frac{3}{4}$
  - Probability of at least three heads come up? $P(E) = 0$
- When tossing a coin until it comes up heads:
  - Probability of coin being tossed at least 3 times?

Cannot determine this by counting because the outcomes are not equiprobable

Probability: sock examples

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock **without replacement**:

- Probability of getting a matching pair?

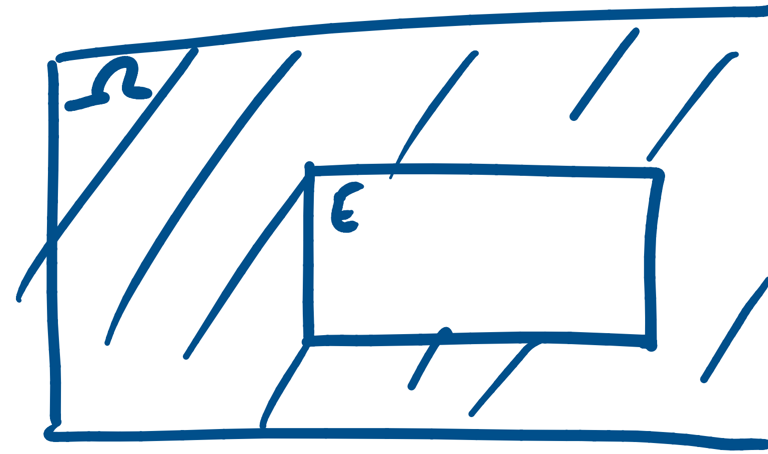
$$P(E) = \frac{2}{12}$$

- Probability of getting a blue sock first or an orange sock second?

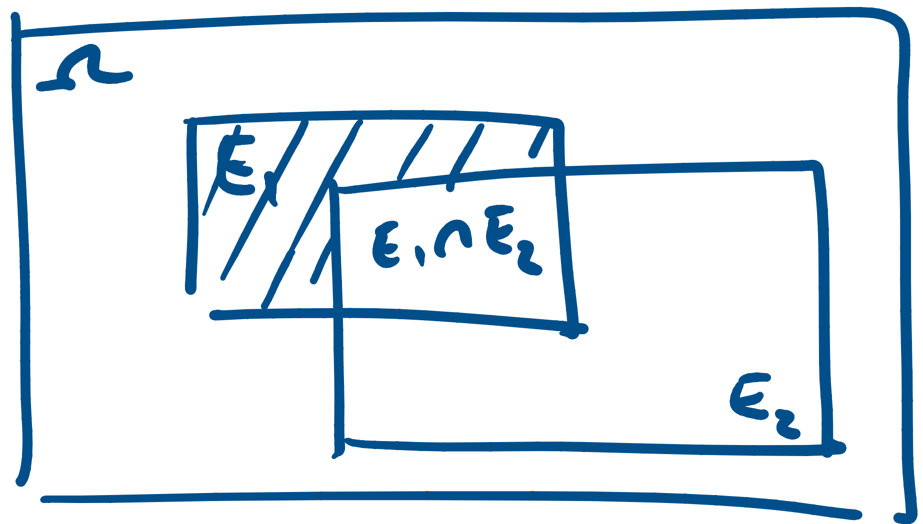
$$P(E) = \frac{7}{12}$$

Properties of probability

- $P(E^c) = 1 - P(E)$

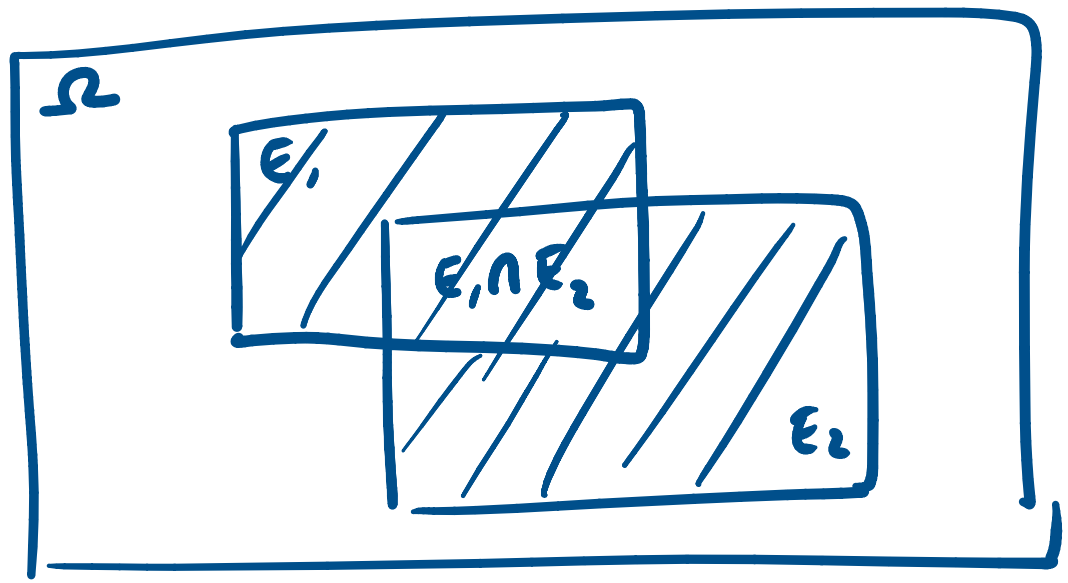


- $P(E_1 - E_2) = P(E_1) - P(E_1 \cap E_2)$



More properties of probability

- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$



- $P(E_1 \cup E_2 \cup E_3)$

$$
\begin{aligned}
&= P(E_1) + P(E_2) + P(E_3) \\
&\quad -P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) \\
&\quad +P(E_1 \cap E_2 \cap E_3)
\end{aligned}
$$

Complement property: sock example

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock **without replacement**:

- Probability of $\underbrace{\text{getting a blue sock first or an orange sock second}}_{E}$?

E^c = (O or W sock first) AND (B or W sock second)

$$P(E) = 1 - P(E^c) = 1 - \frac{5}{12} = \frac{7}{12}$$

Union property: sock example

When drawing 2 socks one-at-a-time from a bag containing 2 blue socks, 1 orange sock and 1 white sock **without replacement**:

- Probability of $\underbrace{\text{getting a blue sock first}}_{E_1}$ or an $\underbrace{\text{orange sock second}}_{E_2}$?

$P(E_1) = \frac{6}{12}$ $\quad P(E_2) = \frac{3}{12}$

$P(E_1 \cap E_2) = \frac{2}{12}$ $\quad P(E_1 \cup E_2) = \frac{6}{12} + \frac{3}{12} - \frac{2}{12} = \frac{7}{12}$