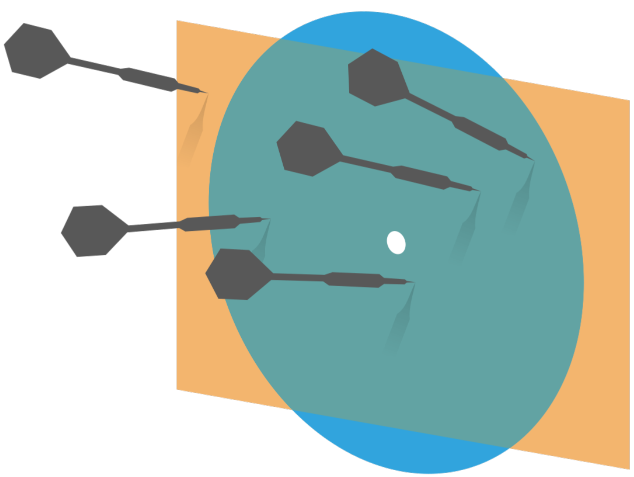
Probability and Statistics for Computer Science



Credit: wikipedia

"It's straightforward to link a number to the outcome of an experiment. The result is a **Random variable**." ---Prof. Forsythe

Random variable is a function, it is not the same as in **X = X+1**

Which is larger?

① The probability of drawing hands of 5-cards that have no pairs.

(no replacement)

② 0.5

> 0.5

A. ① is larger   B. ① is larger

5 permu

$|\Omega| = {}^{52}P_5$

$= 52 \times 51 \times 50 \times 49 \times 48$

$|E| \to \dfrac{52 \times 48 \times 44 \times 40 \times 36}{|\Omega|}$

Last time

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Conditional probability

* Product rule of joint prob.

* Bayes rule

$$P(A|B) = P(A)$$

* Independence

$$P(A \cap B) = P(A)P(B)$$

Objectives

Random Variable

Interface btw Math. CS and the world

* Definition
* Probability distribution
  PDF, CDF
* Conditional probability distri.

Random numbers

- Amount of money on a bet
- Age at retirement of a population
- Rate of vehicles passing by the toll
- Body temperature of a puppy in its pet clinic
- Level of the intensity of pain in a toothache

Degree of a node in a network

...

Random variable as vectors

field

Brain imaging of Human emotions

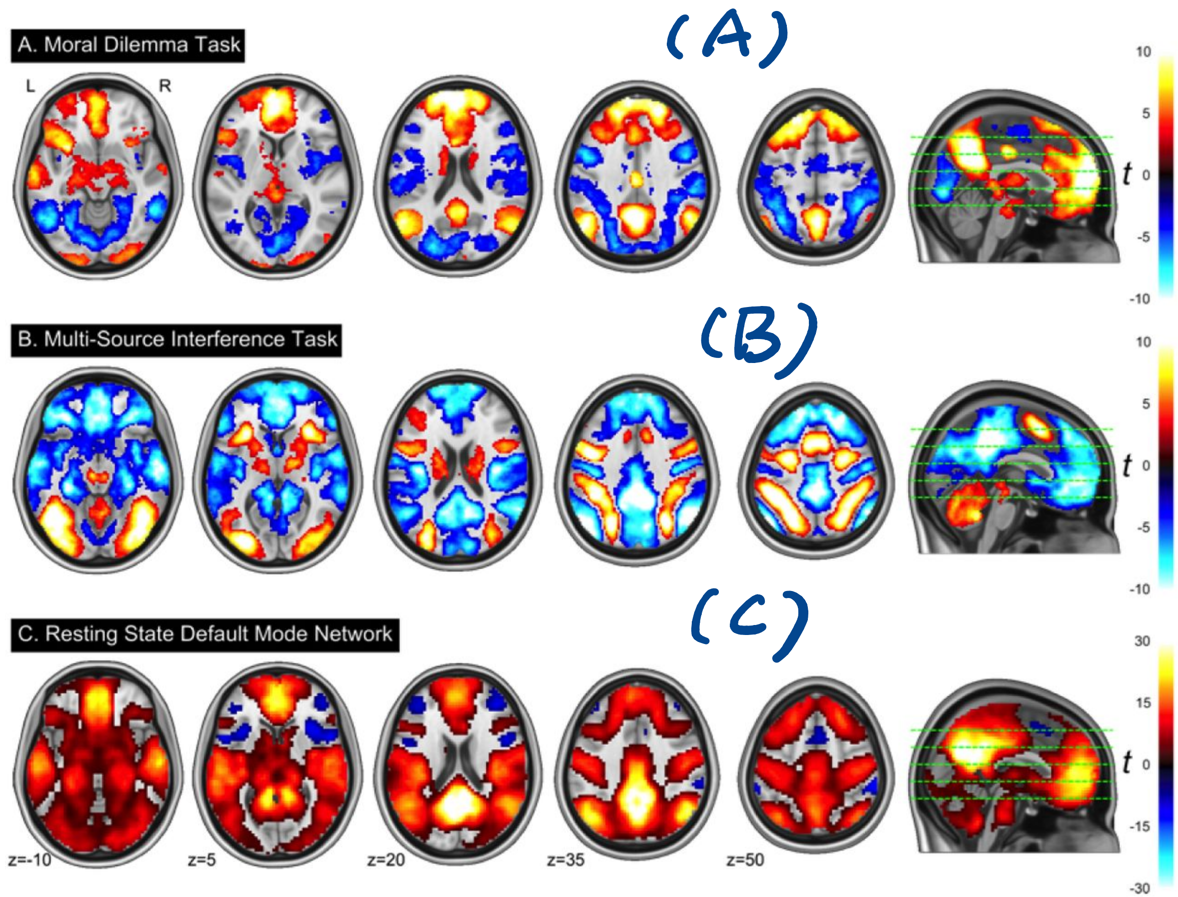
A) Moral conflict

B) Multi-task

C) Rest

time

$(x, y, z, t, \dots)$



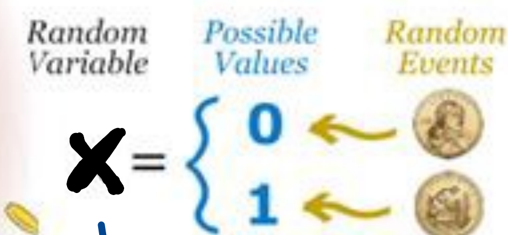
A. McDonald et al. NeuroImage doi: 10.1016/j.neuroimage.2016.10.048

Random variables

A random variable maps all outcomes (ω) to Numbers (X), so

it's a function!!

Bernoulli:



ω is tail

ω is head

$X(\omega)$

$p(\text{tail}) = \frac{1}{2}$

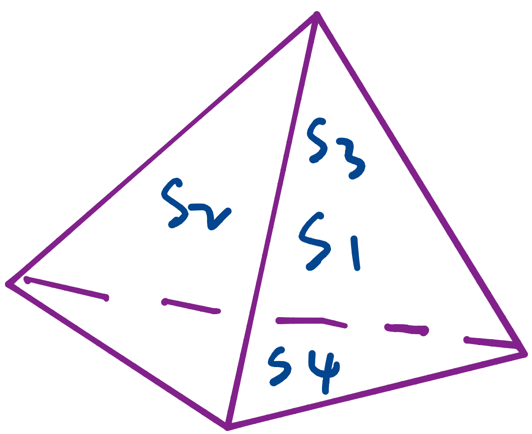
Random variables

- The values of a random variable can be either **discrete**, **continuous** or **mixed**.

Discrete Random variables

- The range of a discrete random variable is a countable set of real numbers.

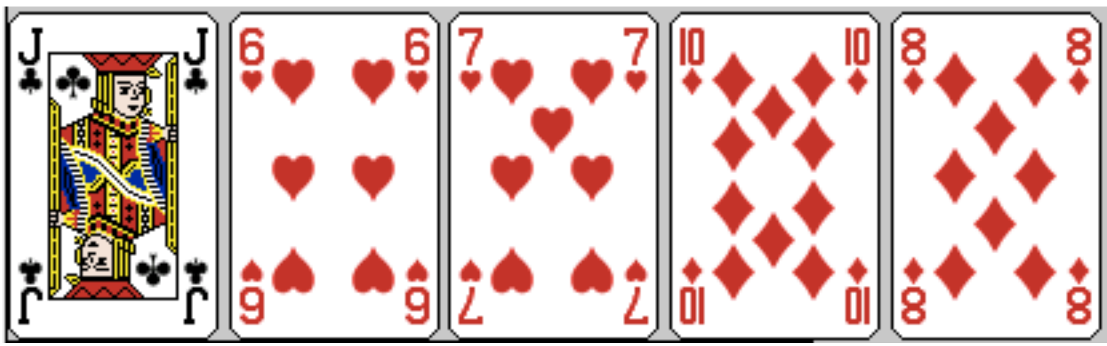
4-die



ω		$X(\omega)$	or	
s_1	$\rightarrow$	1		1
s_2	$\rightarrow$	2		1
s_3	$\rightarrow$	3		1
s_4	$\rightarrow$	4		5
				...

Random Variable Example

- **Number of pairs in a hand of 5 cards**

$\omega =$ 

$X(\omega) = ?$ 0

- Let a single outcome be the hand of 5 cards
- Each outcome maps to values in the set of numbers {0, 1, 2}

0, 1, 2

Random Variable Example

- **Number of pairs in a hand of 6 cards**
  - Let a single outcome be the hand of 6 cards
  - What is the range of values of this random variable?

0, 1, 2, 3

Q: Random Variable

- If we roll a 3-sided fair die, and define random variable *U, such that*

$$U(\omega) = \begin{cases} -1 & \omega \text{ is side } 1 \\ 0 & \omega \text{ is side } 2 \\ 1 & \omega \text{ is side } 3 \end{cases}$$

what is the range of

$X = U^2$ can take

A. {-1, 0, 1}

B. {0, 1}

Three important facts of Random variables

- Random variables have **probability functions**
- Random variables can be **conditioned** on events or other random variables
- Random variables have **averages**

Random variables have **probability functions**

- Let X be a random variable
- The set of outcomes $\{\omega \in \Omega \text{ s.t. } X(\omega) = x_0\}$

  is an event with probability

$$P(X = x_0) = P(\{\omega \text{ s.t. } X(\omega) = x_0\})$$

X is the random variable

x_0 is any unique instance that X takes on

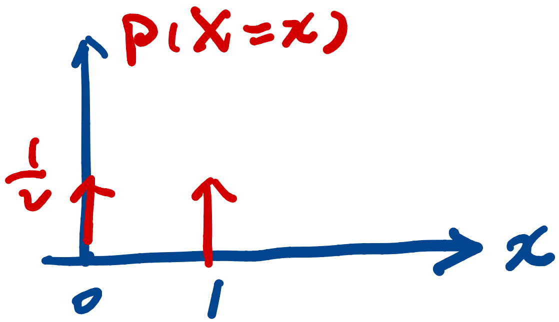
Probability Distribution

- $P(X = x)$ is called the probability distribution for all possible $\boldsymbol{x}$
- $P(X = x)$ is also denoted as $P(x)$ or $p(x)$
- $P(X = x) \geq 0$ for all values that X can take, and is 0 everywhere else
- The sum of the probability distribution is 1

$$\sum_x P(x) = 1$$

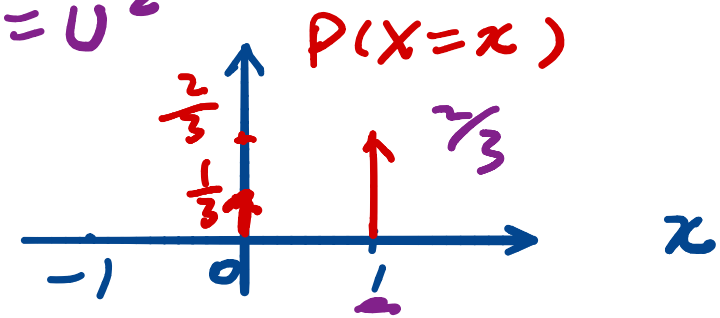
Examples of Probability Distributions

$$X(\omega) = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$$

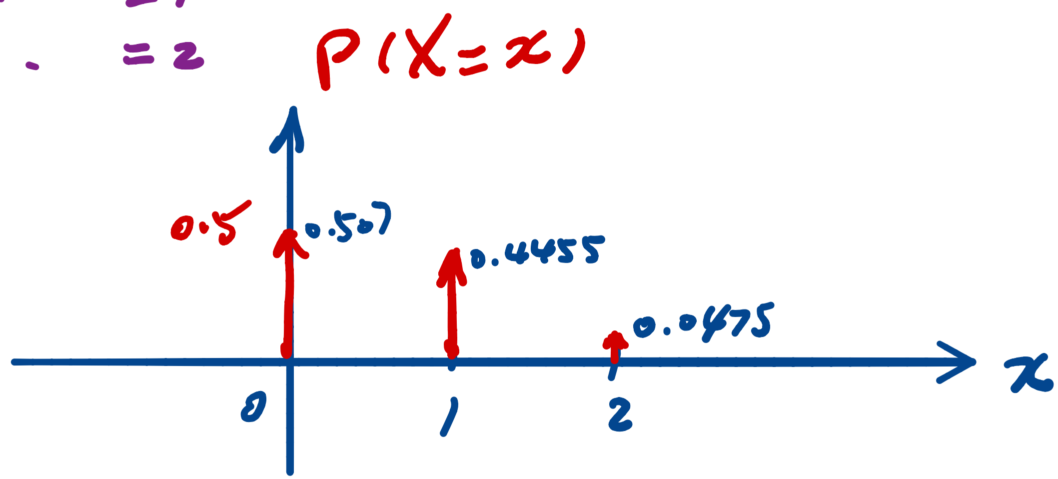


$$U(\omega) = \begin{cases} -1 & \text{side 1} \\ 0 & \text{side 2} \\ 1 & \text{side 3} \end{cases} \quad \frac{1}{3} \;\vdots\; 1$$

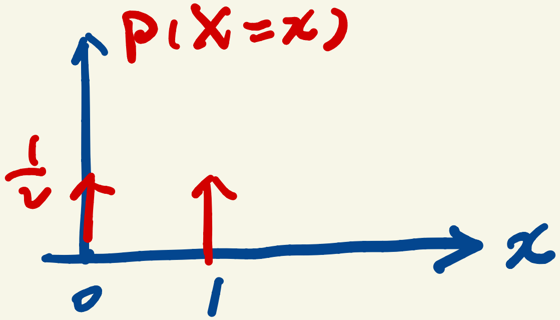
$$X(\omega) = U^2$$



$$X(\omega) = \begin{cases} 0 & \text{\# of pairs} = 0 \\ 1 & \quad \ldots \quad = 1 \\ 2 & \quad \ldots \quad = 2 \end{cases}$$



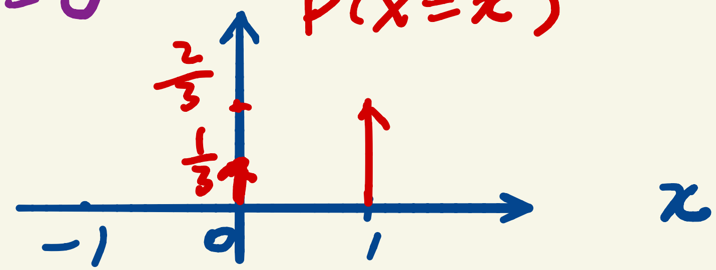
$$X(\omega)=\begin{cases}1 & \text{head}\\ 0 & \text{tail}\end{cases}$$



$$P(X=x)=\begin{cases}\frac{1}{2} & x=0\\ \frac{1}{2} & x=1\\ 0 & \text{otherwise}\end{cases}$$

$$U(\omega)=\begin{cases}-1 & \text{side } 1\\ 0 & \text{side } 2\\ 1 & \text{side } 3\end{cases}$$

$$X(\omega)=U^2$$



$$P(X=x)=\begin{cases}\frac{1}{3} & x=0\\ \frac{2}{3} & x=1\\ 0 & \text{otherwise}\end{cases}$$

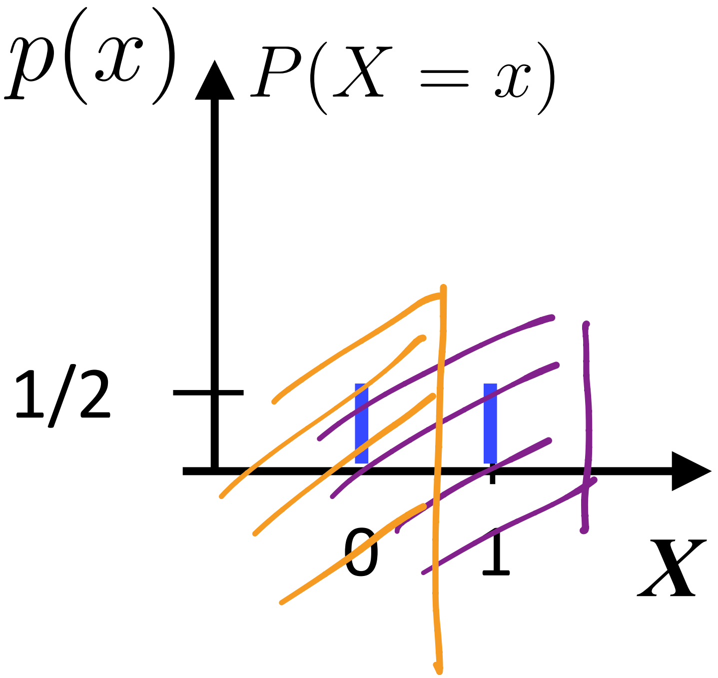
Cumulative distribution

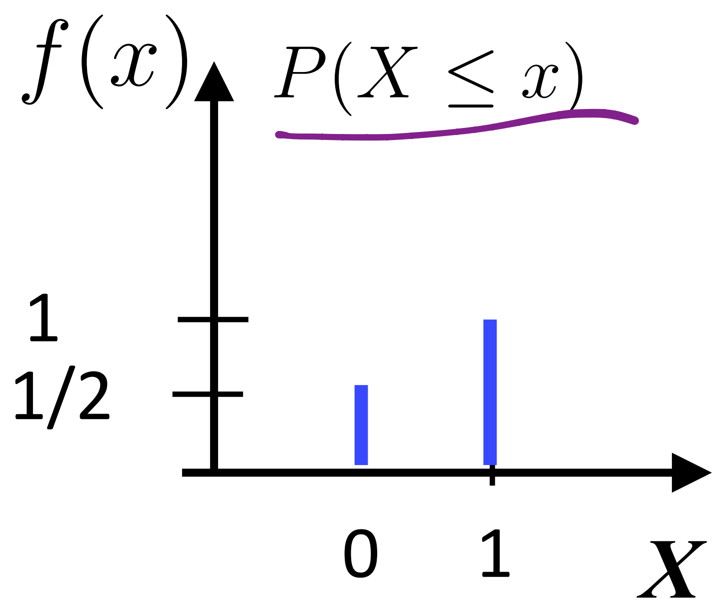
- $P(X \leq x)$ is called the cumulative distribution function of X
- $P(X \leq x)$ is also denoted as $f(x)$
- $P(X \leq x)$ is a non-decreasing function of x

Probability distribution and cumulative distribution

- Give the random variable ***X***,

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$





Q. What is the value?

A biased four-sided die is rolled once. Random variable X is defined to be the down-face value.

$$P(X=x) = \begin{cases} \frac{x}{10} & x = 1, 2, 3, 4 \\ 0 & \text{all others} \end{cases}$$

$\rightarrow P(X \leq 4)$

A) 0.1

B) 0.3

C) 0.2

D) 0.6

$\boxed{\text{E)}}$ 1

Functions of Random Variables

$$U = \begin{cases} -1 & \text{side } 1 \\ 0 & \cdots\ 2 \\ 1 & \ 3 \end{cases}$$

$$X = U^2$$

$$X = X_1 + X_2 + \cdots$$

Q. Are these random variables the same?

$X(\omega) = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$ $\quad Y(\omega) = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$

Bernoulli: RV.

$0 \times 2 = 0$

$1 \times 2 = 2$

$U = \underline{2X}, \quad V = X + Y$

X	Y	V
0	0	0
0	1	1
1	0	1
1	1	2

A) U and V are the same

$\boxed{B)}$ U and V are <u>Not</u> the same.

$U \to \{0, 2\}$ $\quad V \to \{0, 1, 2\}$

Function of random variables: die example

Roll 4-sided fair die twice.

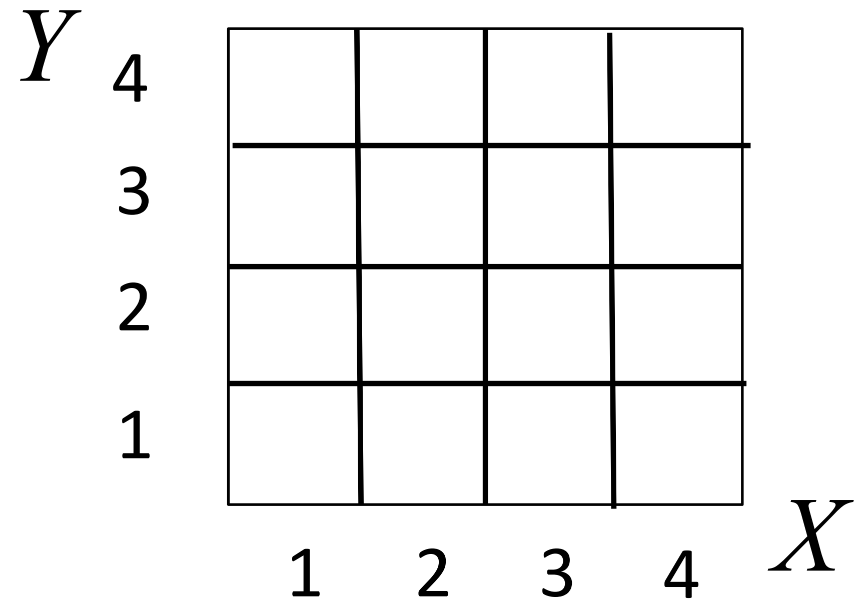
Define these random variables:

X, the values of 1st roll

Y, the values of 2nd roll

Sum $S = X + Y$

Difference $D = X - Y$



$4 \times 4 = 16$

Size of Sample Space = ?

Random variable: die example

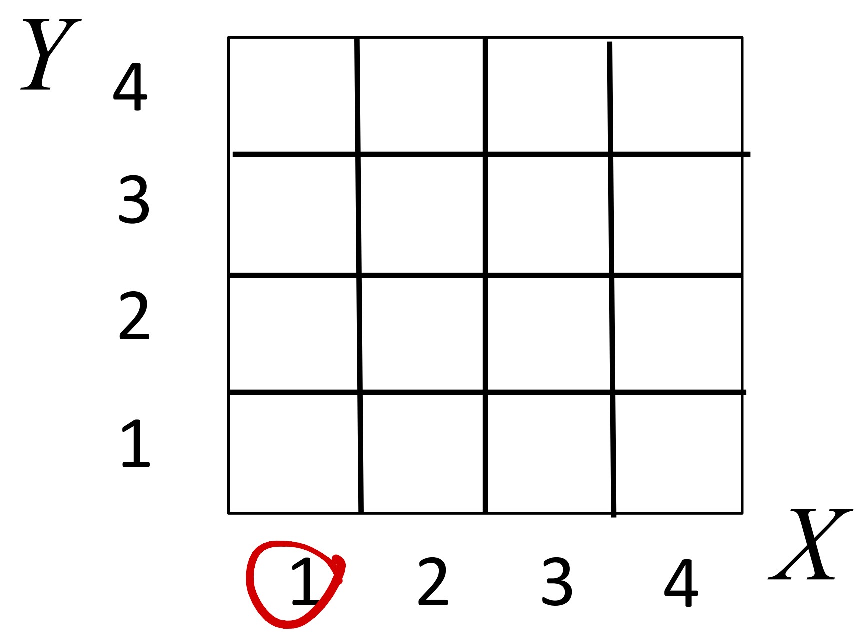
Roll 4-sided fair die twice.

$P(X = 1) = \frac{1}{4}$

$P(Y \leq 2) = \frac{1}{2}$

$P(S = 7)$

$P(D \leq -1)$



Size of Sample Space = 16

Random variable: die example

$S = X + Y$

Y \ X	1	2	3	4
4	5	6	7	8
3	4	5	6	7
2	3	4	5	6
1	2	3	4	5

$D = X - Y$

Y \ X	1	2	3	4
4	-3	-2	-1	0
3	-2	-1	0	1
2	-1	0	1	2
1	0	1	2	3

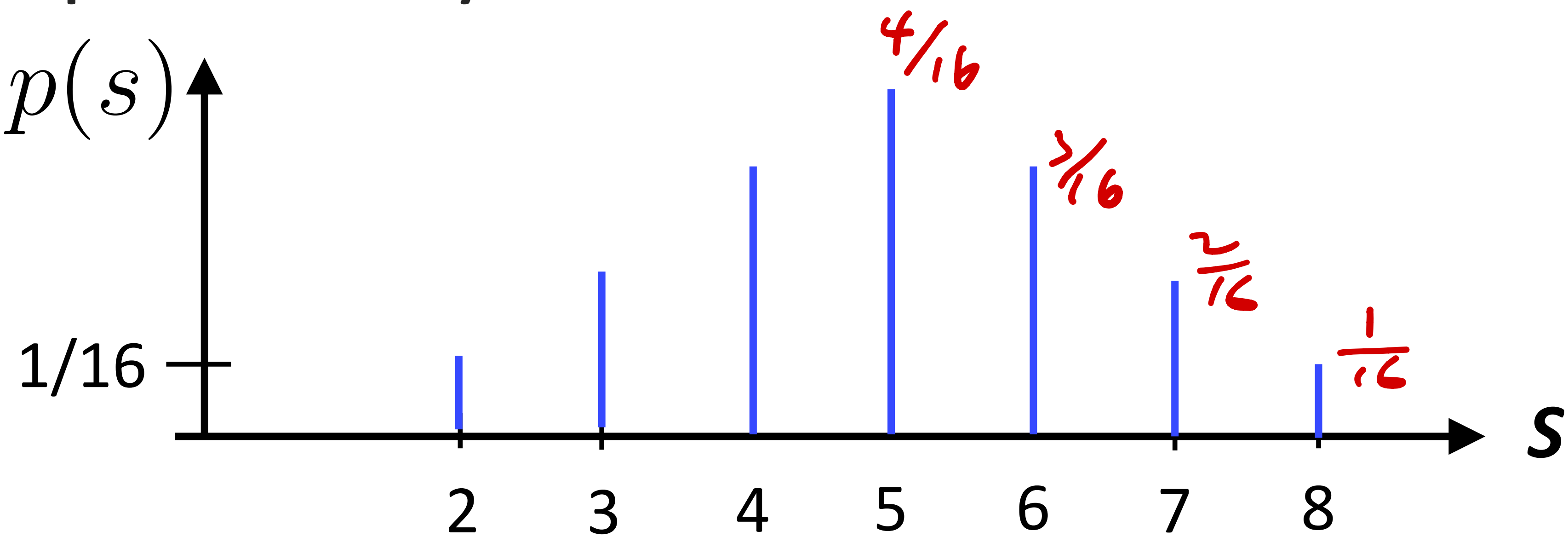
$P(\omega \text{ s.t. } S(\omega)=7)$

$$P(S=7) = \frac{2}{16}$$

$$P(D \leq -1) = \frac{6}{16}$$

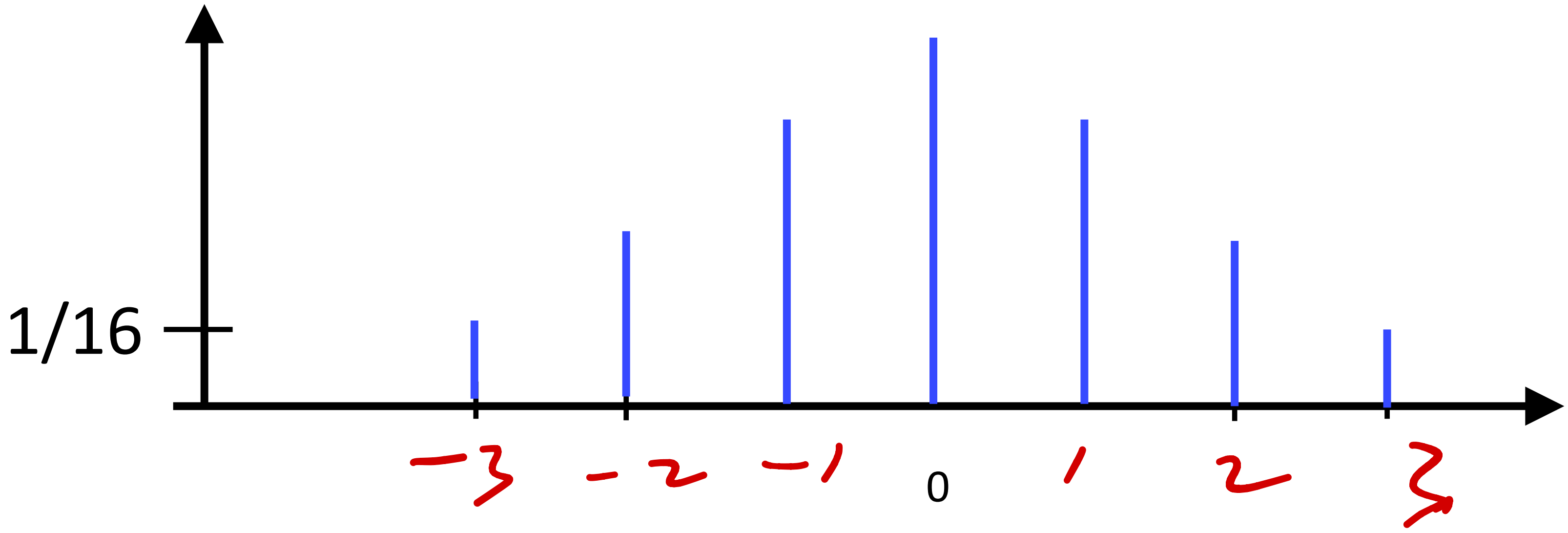
Probability distribution of the sum of two random variables

- Give the random variable ***S*** *in the 4-sided die, whose range is* {2,3,4,5,6,7,8}, probability distribution of ***S.***



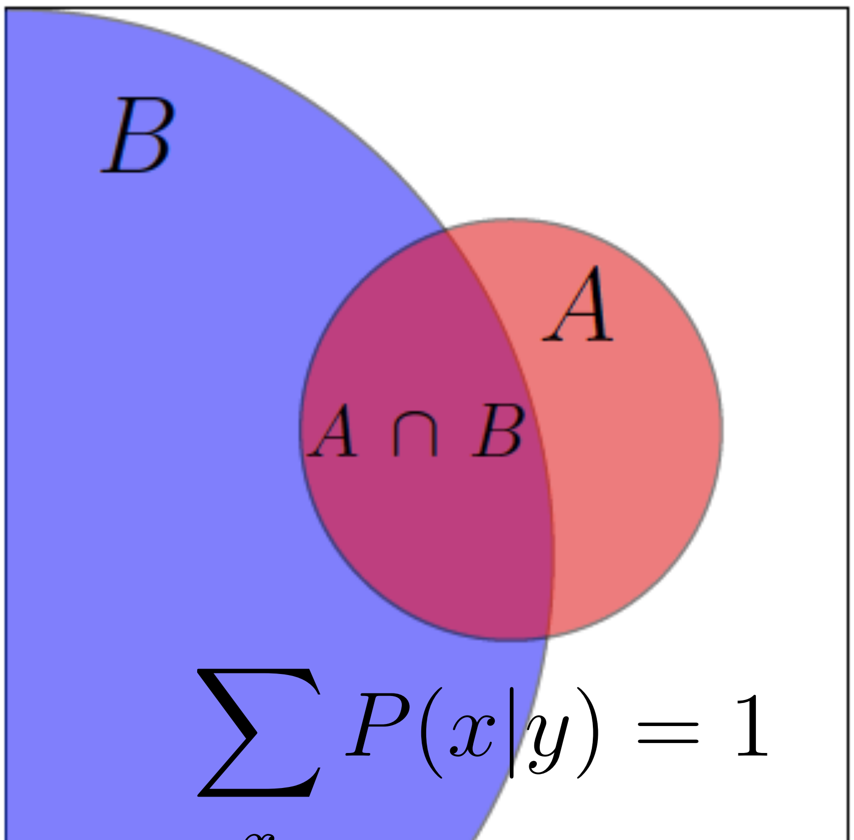
Probability distribution of the difference of two random variables

- Give the random variable ***D = X-Y****, what is the probability distribution of* ***D****?*



Conditional Probability

- The probability of ***A*** given ***B***



$$\sum_x P(x|y) = 1$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B) \neq 0$$

The “Size” analogy

Credit: Prof. Jeremy Orloff & Jonathan Bloom

Conditional probability distribution of random variables

- The conditional probability distribution of X *given* Y is

$$P(x|y) = \frac{P(x,y)}{P(y)} \qquad P(y) \neq 0$$

$$P(x,y) = P(X=x \cap Y=y)$$

$$P(y) = P(Y=y)$$

$$P(x|y) = P(X=x|Y=y)$$

Get the marginal from joint distri.

- We can recover the individual probability distributions from the joint probability distribution

$$P(x) = \sum_{y} P(x,y)$$

$$P(y) = \sum_{x} P(x,y)$$

$p(x,y) = \underline{p(y|x)\,p(x)}$

$\sum_{y} p(x,y)$

$= \sum_{y} p(y|x)\,p(x)$

$= p(x) \underline{\sum_{y} p(y|x)}$ 1

$= p(x)$

Joint probabilities sum to 1

- The sum of the joint probability distribution

$$\sum_y \sum_x \underline{P(x,y)} = 1$$

Ω

Joint Probability Example

- Tossing a coin twice, we define random variable $X\ and\ Y$ for each toss.

$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

Joint probability distribution example

$P(x, y)$

		X: 0	1
Y	0	$\frac{1}{4}$	$\frac{1}{4}$
	1	$\frac{1}{4}$	$\frac{1}{4}$

$P(y)$

$\frac{1}{2}$
$\frac{1}{2}$

$P(x)$

$\frac{1}{2}$	$\frac{1}{2}$

Joint Probability Example

Now we define Sum ***S*** = $X + Y$, Difference ***D*** = $X - Y$. ***S*** takes on values {0,1,2} and ***D*** takes on values {-1, 0, 1}

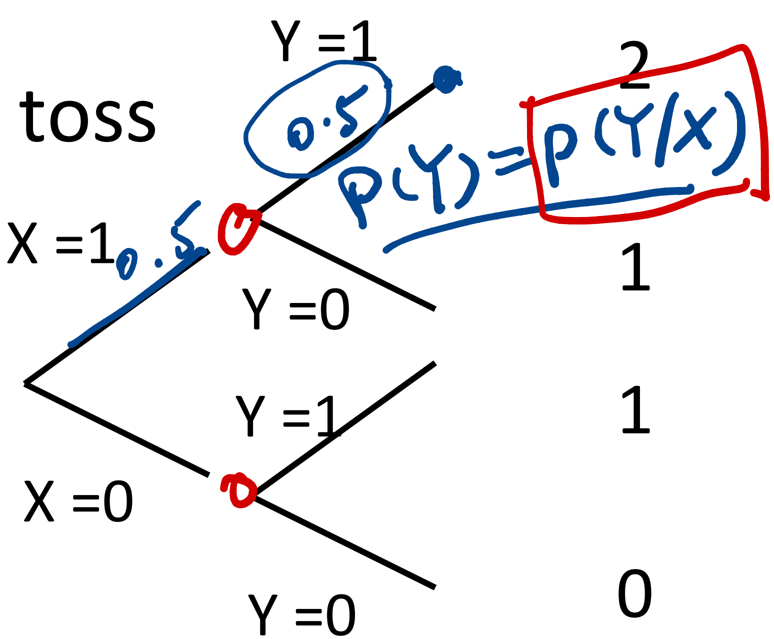
$$X(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

$$Y(\omega) = \begin{cases} 1 & outcome\ of\ \omega\ is\ head \\ 0 & outcome\ of\ \omega\ is\ tail \end{cases}$$

Joint Probability Example

2nd toss

1st toss



S=s D=d

S	D	P(s, d)
		$P(S=s, D=d) = \frac{1}{2} \times \frac{1}{2}$
2	0	
1	1	$\frac{1}{2} \times \frac{1}{2}$
1	-1	$\frac{1}{4}$
0	0	$\frac{1}{4}$

$P(A \cap B) = P(B|A)\,P(A) = P(A|B)\,P(B) = P(A), P(B)$

Suppose coin is fair, and the tosses are independent

Joint probability distribution example

$P(s, d)$

S \ D	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{4}$	0	$\frac{1}{4}$
2	0	$\frac{1}{4}$	0

$P(s)$

0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$P(d)$

-1	0	1
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$P(S) = \sum_{D} P(S, D)$$

Independence of random variables

- Random variable $X\ and\ Y$ are independent if

$$P(x,y) = P(x)P(y)\ \boxed{for\ all\ x\ and\ y}$$

- In the previous coin toss example
  - Are X and Y independent?
  - Are ***S*** and ***D*** independent?

$P(S, D) = P(S)\,P(D)$ for all s, d.

Joint probability distribution example

$P(x, y)$

		X = 0	X = 1
Y	0	$\frac{1}{4}$	$\frac{1}{4}$
	1	$\frac{1}{4}$	$\frac{1}{4}$

$P(y)$

0	$\frac{1}{2}$
1	$\frac{1}{2}$

$P(x)$

0	1
$\frac{1}{2}$	$\frac{1}{2}$

Joint probability distribution example

$P(s, d)$

S \ D	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{4}$	0	$\frac{1}{4}$
2	0	$\frac{1}{4}$	0

$P(s)$

s	$P(s)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$P(d)$

d	-1	0	1
$P(d)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$s = 1, d = 0$

$P(s=1, d=0) = 0$

$P(s=1)\,P(d=0) = \frac{1}{4}$

S, D are Not indept.

Joint probability distribution example

$P(s, d)$

S \ D	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{4}$	$\boxed{0}$	$\frac{1}{4}$
2	0	$\frac{1}{4}$	0

$P(s)$

0	$\frac{1}{4}$
1	$\boxed{\frac{1}{2}}$
2	$\frac{1}{4}$

$P(d)$

-1	0	1
$\frac{1}{4}$	$\boxed{\frac{1}{2}}$	$\frac{1}{4}$

$$P(S=1, D=0) \neq P(S=1)P(D=0)$$

Conditional probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	1	0	1
2	0	$\frac{1}{2}$	0

Bayes rule for random variable

- Bayes rule for events generalizes to random variables

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$\boxed{\begin{aligned} P(x|y) &= \frac{P(y|x)P(x)}{P(y)} \\ &= \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)} \end{aligned}}$$

Total Probability

Conditional probability distribution example

$$P(s|d) = \frac{P(s,d)}{P(d)}$$

S \ D	-1	0	1
0	0	$\frac{1}{2}$	0
1	1	0	1
2	0	$\frac{1}{2}$	0

$$P(D=-1|S=1) = \frac{P(S=1|D=-1)P(D=-1)}{P(S=1)} = \frac{1 \times \frac{1}{4}}{\frac{1}{2}}$$

Assignments

- Chapter 4 of the textbook
- Next time: More random variable, Expectations, Variance

Additional References

- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

See you next time

See You!