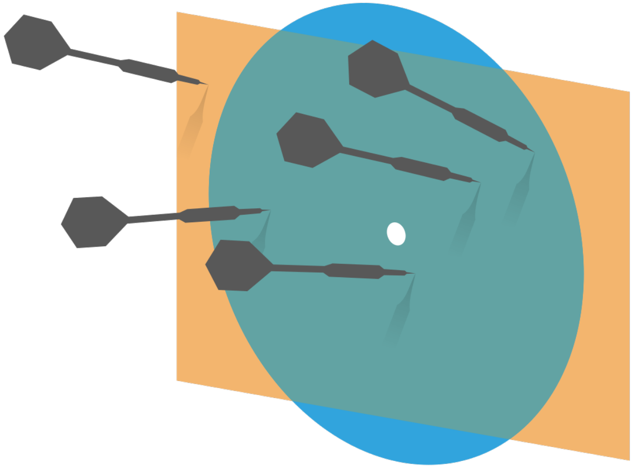
# Probability and Statistics for Computer Science



Credit: wikipedia

"Correlation is not Causation" but Correlation is so beautiful!

Hongye Liu, Teaching Assistant Prof, CS361, UIUC, 9.1.2020

* Please use "#" sign in your chat to indicate a formal question or comment.

* please mute your mic to keep the zoom sound quality.

* Please check out the websites of Simulation & Code Notebook in the chat.

# Last time

Location Parameters:

Mean ($\mu$), Median, Mode

Scale Parameters:

Standard deviation ($\sigma$), Interquartile range (iqr)

variance ($\sigma^2$)

Standardizing Data: $\hat{x}_i = \frac{x_i - \mu}{\sigma}$

# Objectives

- Median, Interquartile range, box plot and outlier, Mode & Skew
- Scatter plots, Correlation Coefficient
- Visualizing & Summarizing *relationships*

  Heatmap, 3D bar, Time series plots,

# Median

- ~~To organize the data~~ we first sort it
- Then *if* the number of items N is odd

  median = middle item's value

  *if* the number of items N is even

  median = mean of middle 2 items' values

# Properties of Median

- Scaling data scales the median

$$median(\{k \cdot x_i\}) = k \cdot median(\{x_i\})$$

median = $\underset{\mu}{\operatorname{argmin}} \left( \sum_{i=1}^{N} |x_i - \mu| \right)$

- Translating data translates the median

$$median(\{x_i + c\}) = median(\{x_i\}) + c$$

# Percentile

- $k^{th}$ percentile is the value relative to which k% of the data items have smaller or equal numbers
- Median is roughly the $50^{th}$ percentile

{1, 2, 3, 4, 5, 6, 7, 12}

75th percentile = ? 6 $\neq 75\%$

# Interquartile range

- iqr = (75th percentile) - (25th percentile) $+$ $\geq 0$
- Scaling data scales the interquartile range

$$iqr(\{k \cdot x_i\}) = |k| \cdot iqr(\{x_i\})$$

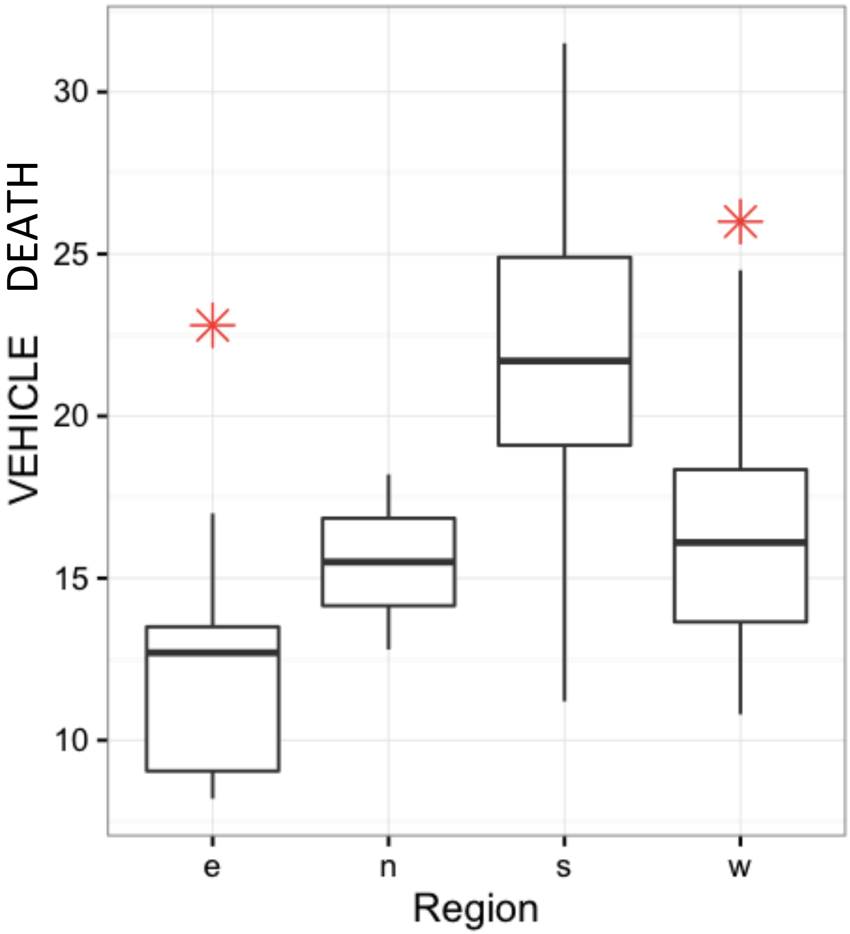
- Translating data does **NOT** change the interquartile range

$$iqr(\{x_i + c\}) = iqr(\{x_i\})$$

# Box plots

- Boxplots
  - Simpler than histogram
  - Good for outliers
  - Easier to use for comparison

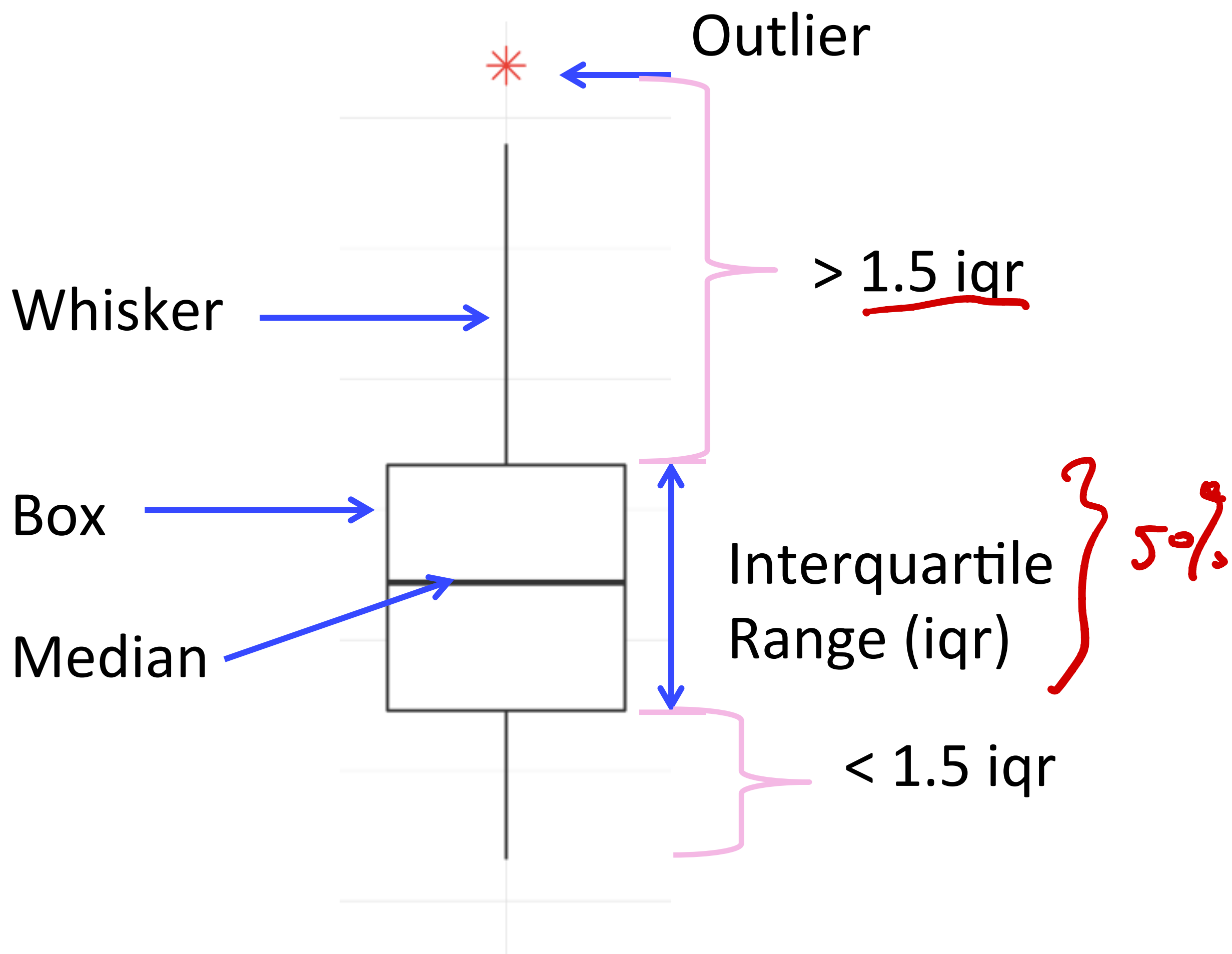
Vehicle death by region




Data from https://www2.stetson.edu/~jrasp/data.htm

# Boxplots details, outliers

- How to define outliers? (the default)



# Q. TRUE or FALSE

mean is more sensitive to outliers than median

A True

B False

# Q. TRUE or FALSE

interquartile range is more sensitive to outliers than std.

A True

$\boxed{B}$ False

# Sensitivity of summary statistics to outliers

- mean and standard deviation are very sensitive to outliers
- median and interquartile range are not sensitive to outliers

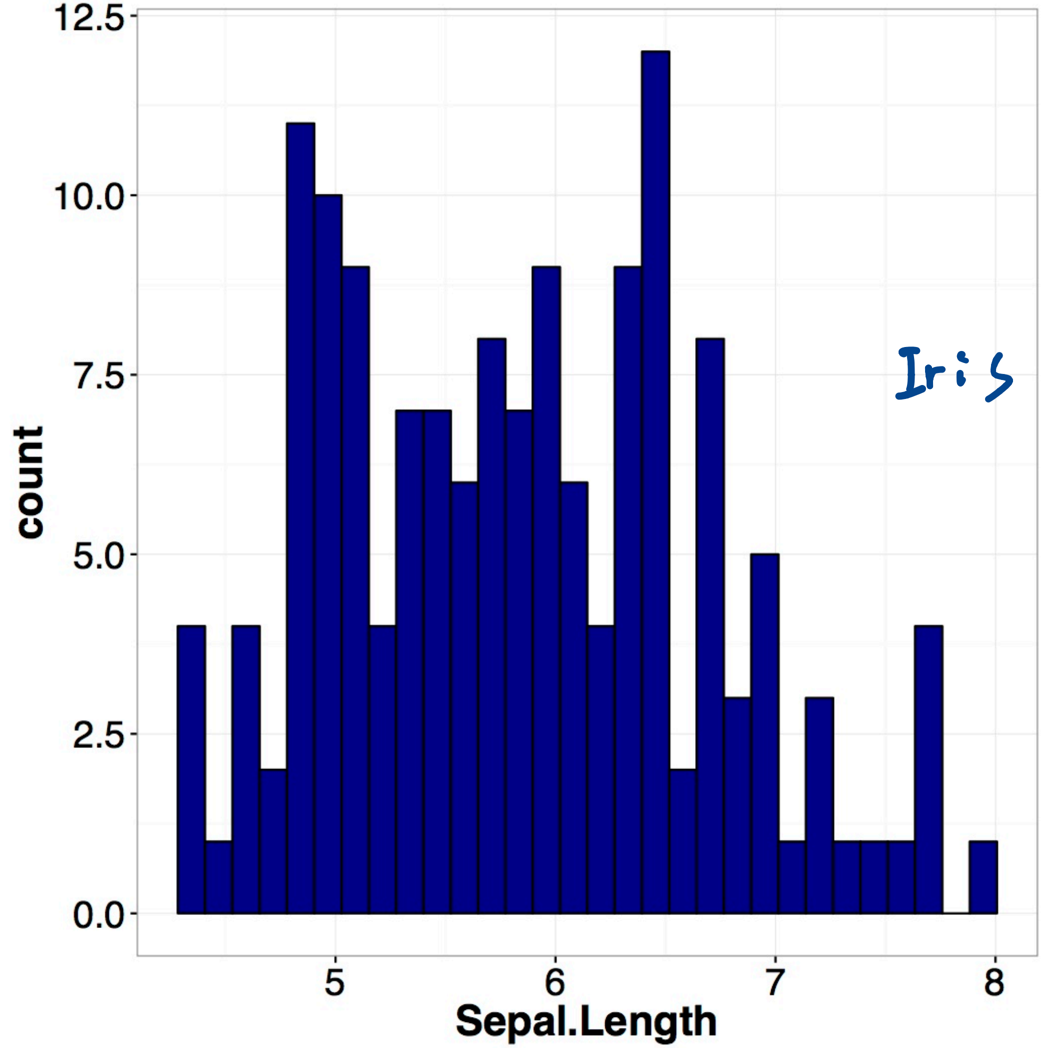
# Modes

- Modes are peaks in a histogram
- If there are more than 1 mode, we should be curious as to why

# Multiple modes

- We have seen the “iris” data which looks to have several peaks

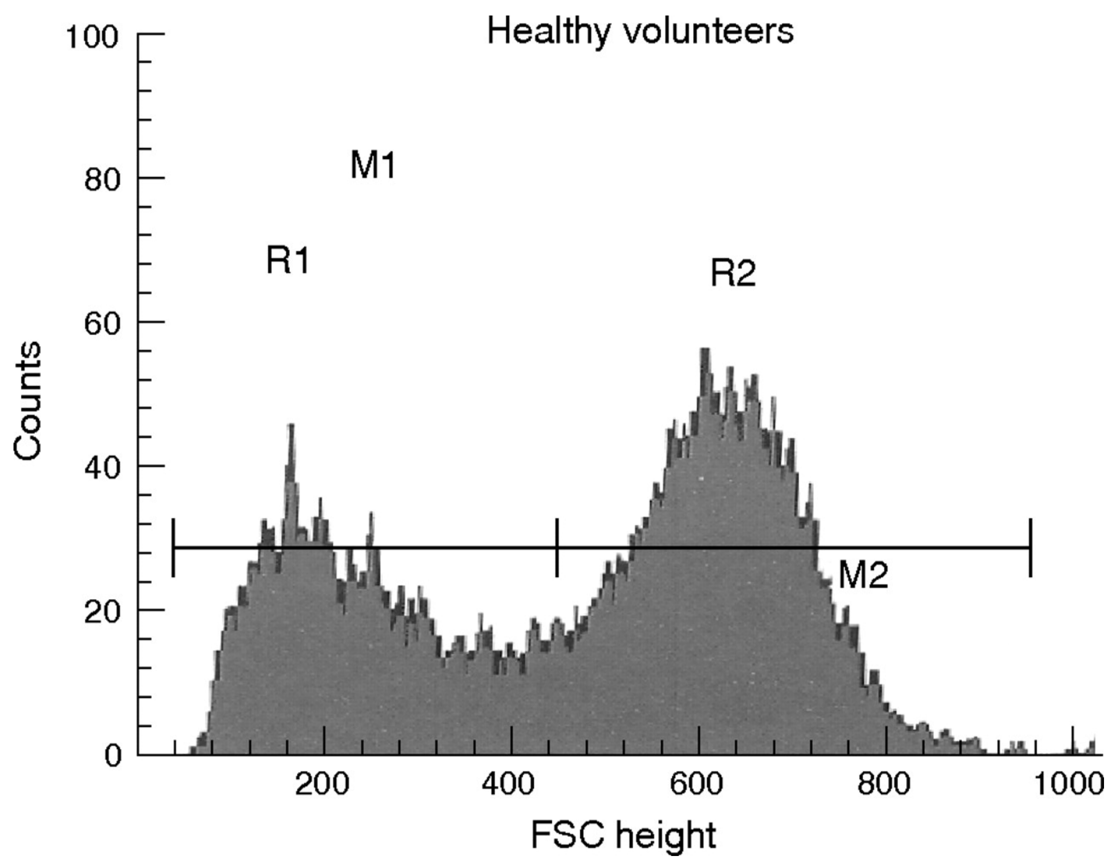
Data: “iris” in R



# Example Bi-modes distribution

- Modes may indicate multiple populations

red blood cell

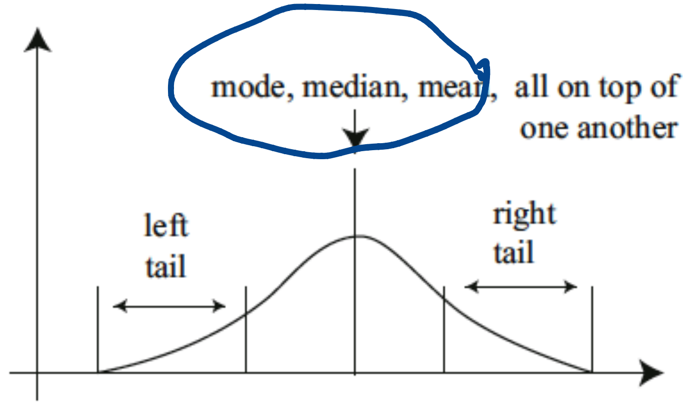



Data: Erythrocyte cells in healthy humans

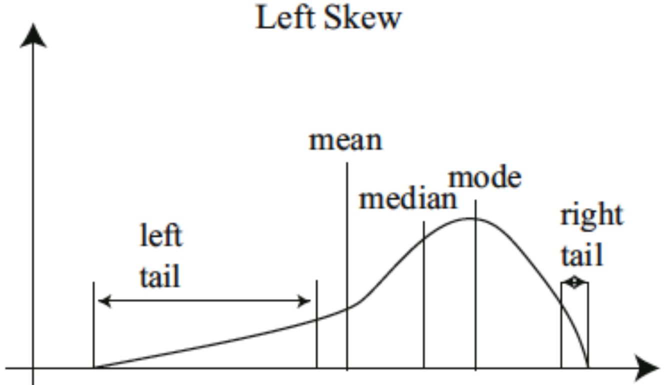
Piagnerelli, JCP 2007

# Tails and Skews

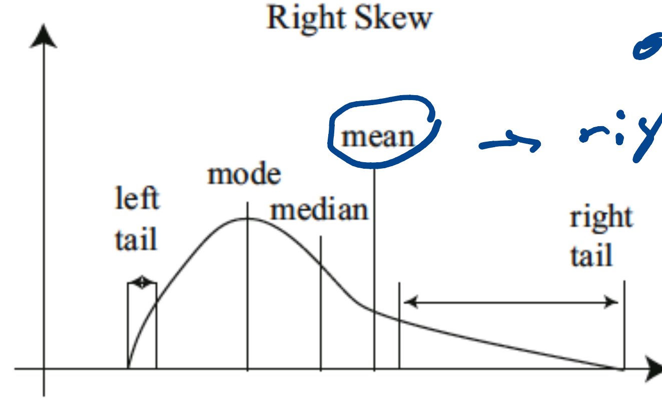
Symmetric Histogram




Left Skew



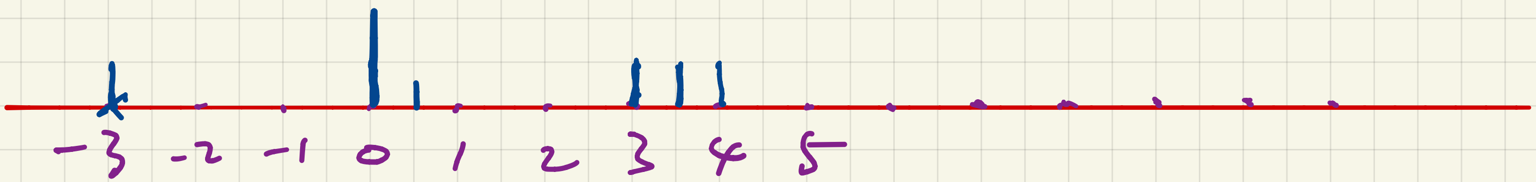

Right Skew




tails
outliers

→ right tail

Credit: Prof.Forsyth



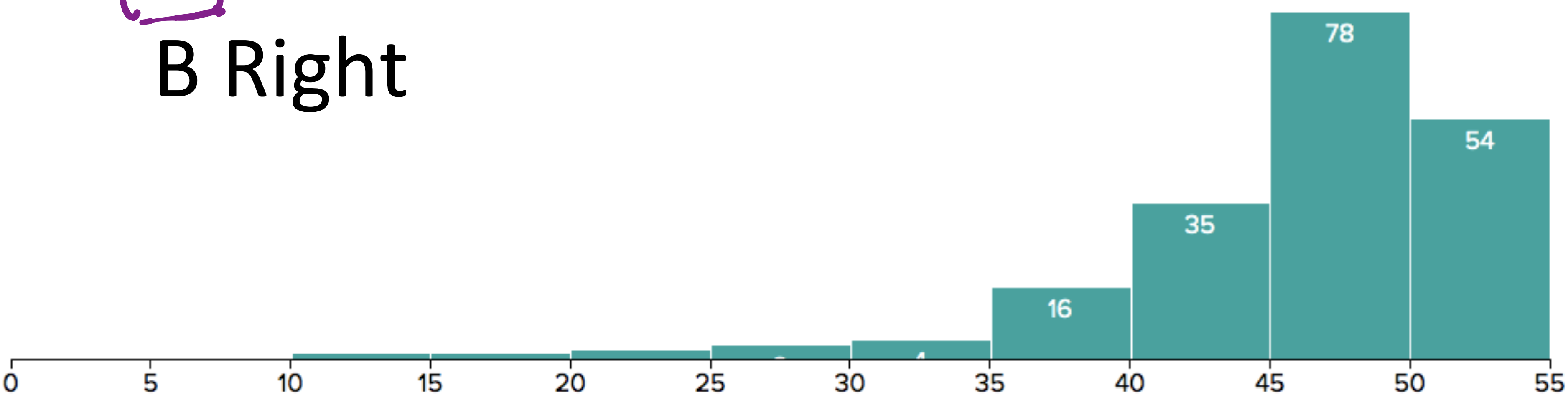

median:

mean

# Q. How is this skewed?

A Left

B Right




mean = ? 46

Median = 47

# Looking at relationships in data

- Finding relationships between features in a data set or many data sets is one of the most important tasks in data analysis

# Relationship between data features

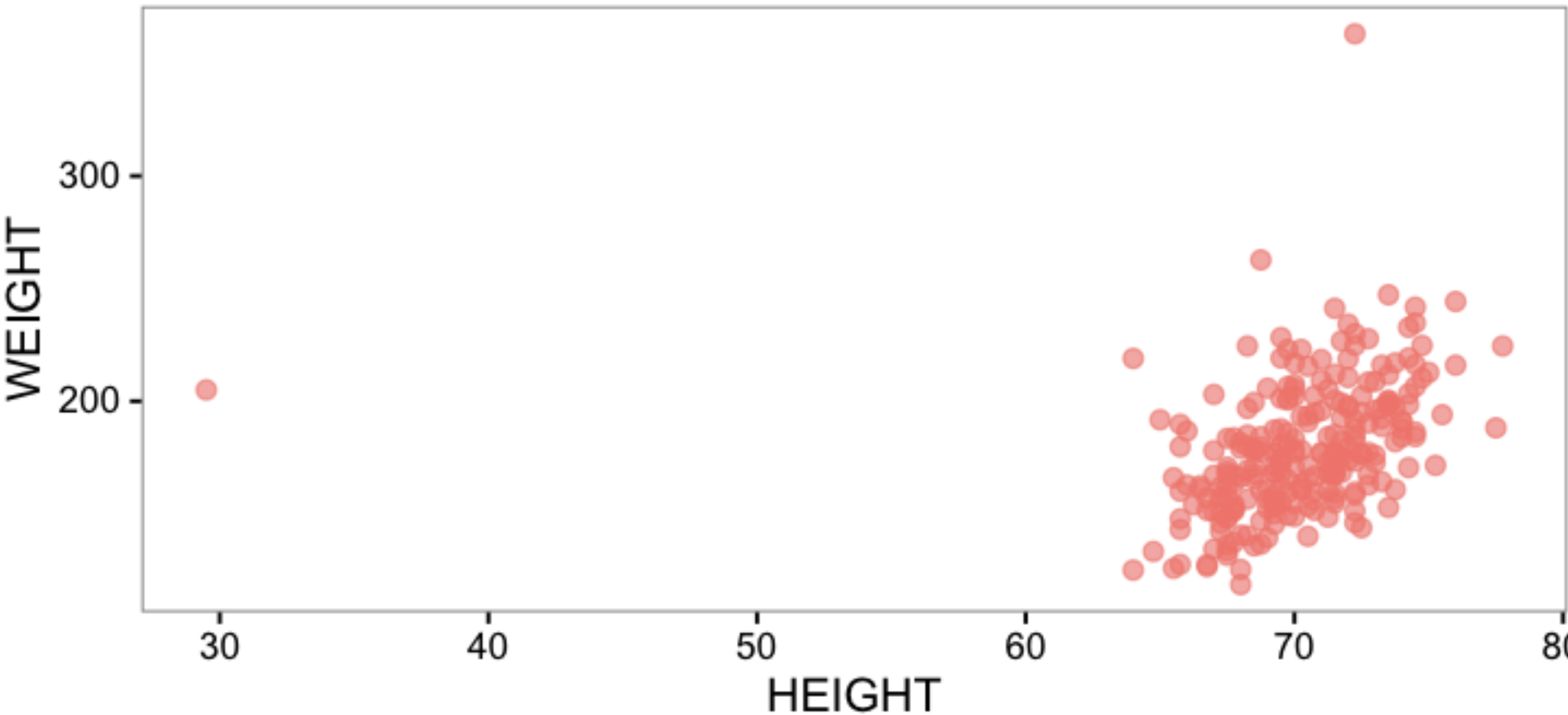
- Example: does the weight of people relate to their height?

| IDNO | BODYFAT | DENSITY | AGE | WEIGHT | HEIGHT |
|---|---|---|---|---|---|
| 1 | 12.6 | 1.0708 | 23 | 154.25 | 67.75 |
| 2 | 6.9 | 1.0853 | 22 | 173.25 | 72.25 |
| 3 | 24.6 | 1.0414 | 22 | 154.00 | 66.25 |
| 4 | 10.9 | 1.0751 | 26 | 184.75 | 72.25 |
| 5 | 27.8 | 1.0340 | 24 | 184.25 | 71.25 |
| 6 | 20.6 | 1.0502 | 24 | 210.25 | 74.75 |
| 7 | 19.0 | 1.0549 | 26 | 181.00 | 69.75 |
| 8 | 12.8 | 1.0704 | 25 | 176.00 | 72.50 |
| 9 | 5.1 | 1.0900 | 25 | 191.00 | 74.00 |
| 10 | 12.0 | 1.0722 | 23 | 198.25 | 73.50 |

- x : HIGHT, y: WEIGHT

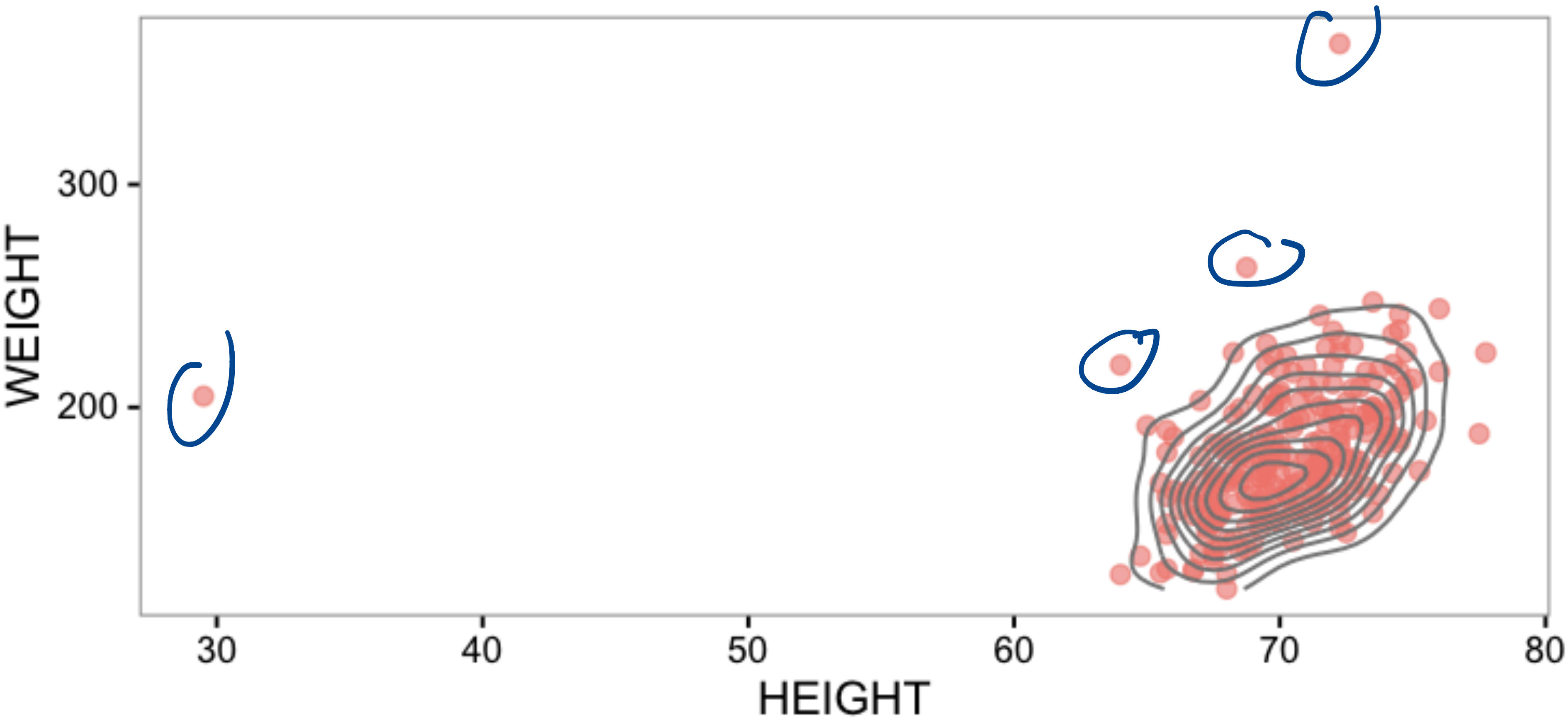
# Scatter plot

- Body Fat data set



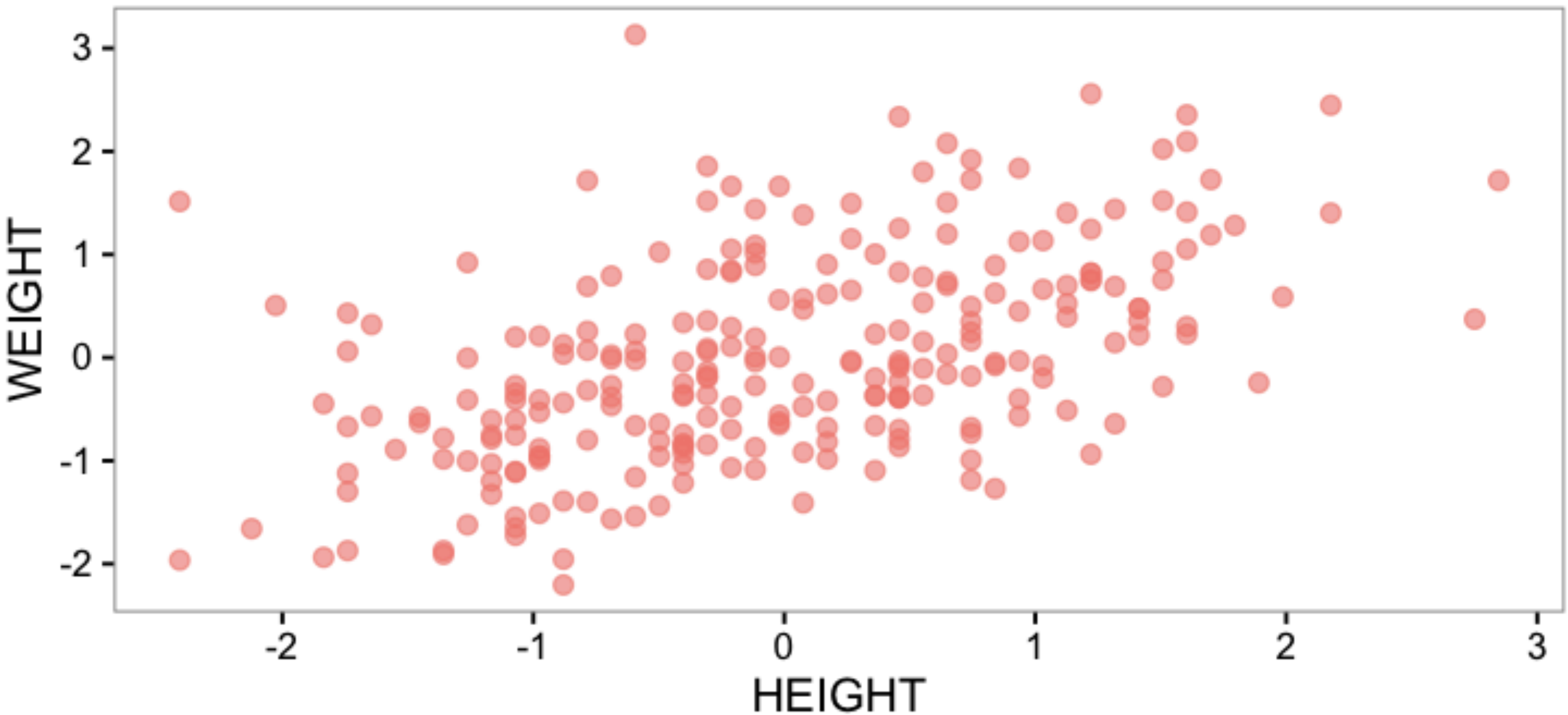
# Scatter plot

- Scatter plot with density

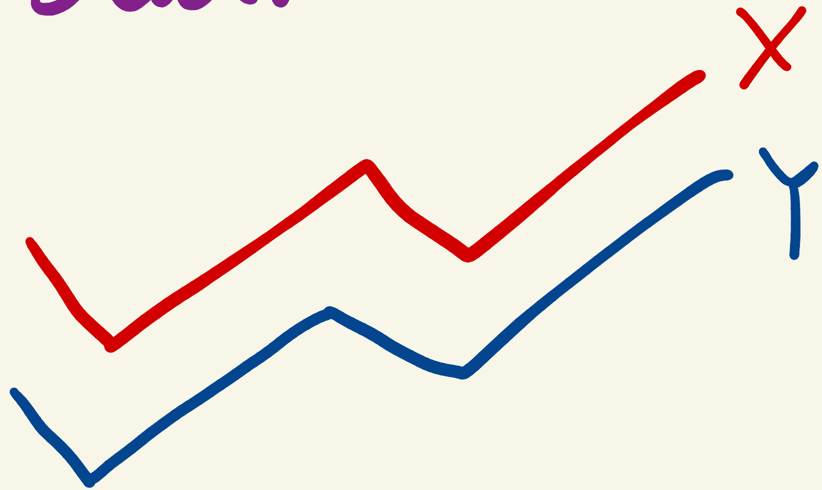


# Scatter plot

- Removed of outliers & standardized



# Correlation

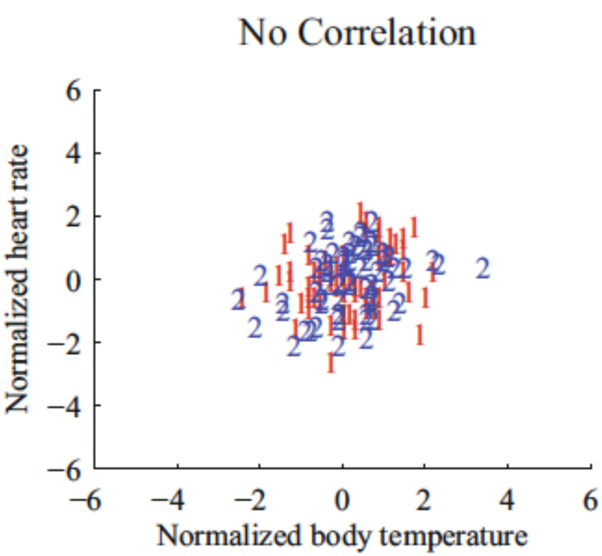


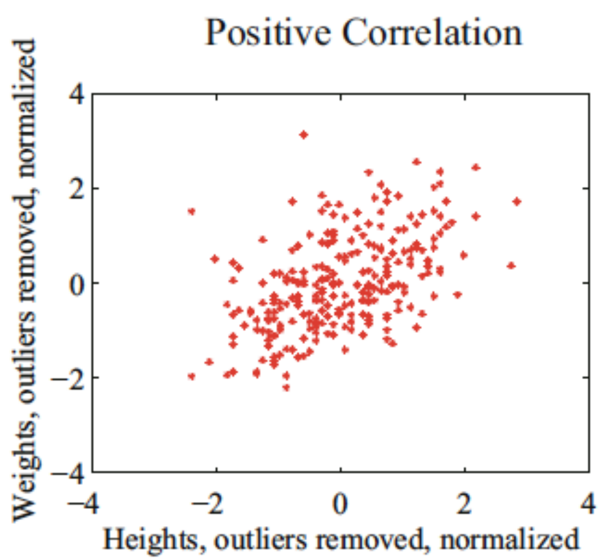

covariance

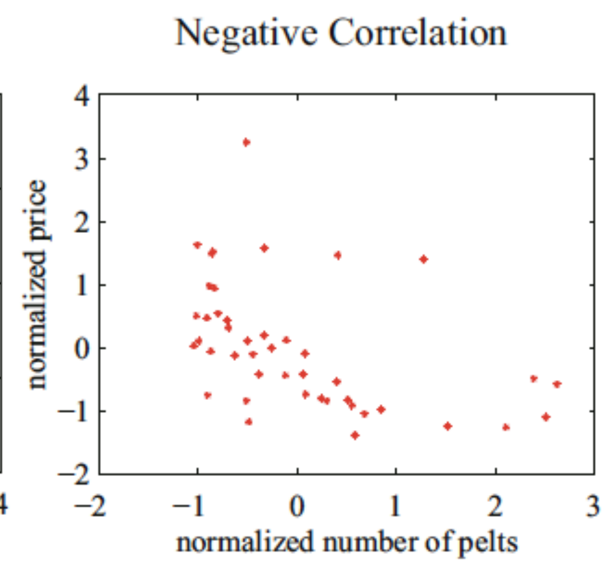
ch. 4
10
13

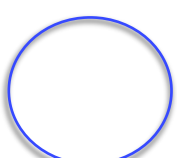
# Correlation seen from scatter plots

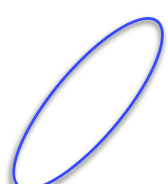
| Zero Correlation | Positive correlation | Negative correlation |
|---|---|---|





Credit: Prof.Forsyth

# What kind of Correlation?

- Line of code in a database and number of bugs
- Frequency of hand washing and number of germs on your hands
- GPA and hours spent playing video games
- earnings and happiness

Credit: Prof. David Varodayan

# Correlation doesn’t mean causation

- Shoe size is correlated to reading skills, but it doesn’t mean making feet grow will make one person read faster.

# Correlation Coefficient

- Given a data set $\{(x_i, y_i)\}$ consisting of items $(x_1, y_1)$ ... $(x_N, y_N)$,
  - Standardize the coordinates of each feature:

$$\widehat{x_i} = \frac{x_i - mean(\{x_i\})}{std(\{x_i\})} \qquad \widehat{y_i} = \frac{y_i - mean(\{y_i\})}{std(\{y_i\})}$$

  - Define the correlation coefficient as:

$$corr(\{(x_i, y_i)\}) = \frac{1}{N}\sum_{i=1}^{N} \widehat{x_i}\widehat{y_i}$$

# Correlation Coefficient

$$\widehat{x_i} = \frac{x_i - mean(\{x_i\})}{std(\{x_i\})} \qquad \widehat{y_i} = \frac{y_i - mean(\{y_i\})}{std(\{y_i\})}$$

$$corr(\{(x_i, y_i)\}) = \frac{1}{N}\sum_{i=1}^{N} \widehat{x_i}\widehat{y_i}$$

$$= mean(\{\widehat{x_i}\widehat{y_i}\})$$

# Q: Correlation Coefficient

- Which of the following describe(s) correlation coefficient correctly?

  A. It's unitless

  B. It's defined in standard coordinates

  C. Both A & B

$$corr(\{(x_i, y_i)\}) = \frac{1}{N}\sum_{i=1}^{N} \widehat{x_i}\widehat{y_i}$$

# A visualization of correlation coefficient

https://rpsychologist.com/d3/correlation/

In a data set $\{(x_i, y_i)\}$ consisting of items $(x_1, y_1)$ … $(x_N, y_N)$,

$corr(\{(x_i, y_i)\}) > 0$ shows positive correlation

$corr(\{(x_i, y_i)\}) < 0$ shows negative correlation

$corr(\{(x_i, y_i)\}) = 0$ shows no correlation

# The Properties of Correlation Coefficient

- The correlation coefficient is symmetric

$$corr(\{(x_i, y_i)\}) = corr(\{(y_i, x_i)\})$$

- Translating the data does **NOT** change the correlation coefficient

# The Properties of Correlation Coefficient

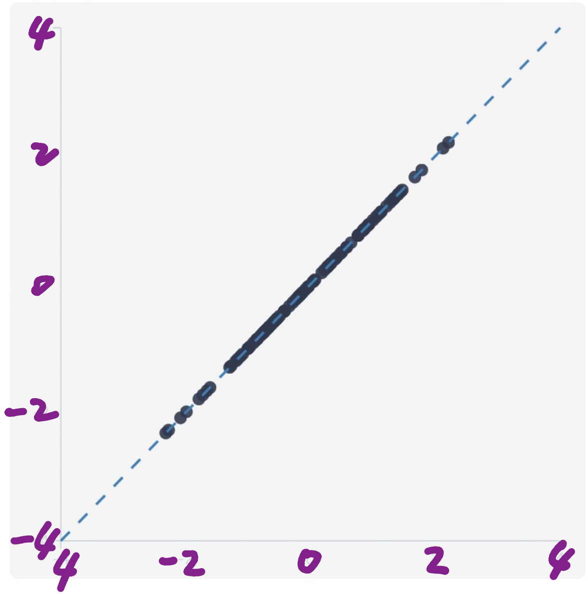
- Scaling the data may change the sign of the correlation coefficient

$$corr(\{(a\ x_i + b,\ c\ y_i + d)\})$$

$$= sign(a\ c) corr(\{(x_i, y_i)\})$$

Correlation is one of the most widely used tools in statistics. The correlation coefficient summarizes the association between two variables. In this visualization I show a scatter plot of two variables with a given correlation. The variables are samples from the standard normal distribution, which are then transformed to have a given correlation by using Cholesky decomposition. By moving the slider you will see how the shape of the data changes as the association becomes stronger or weaker. You can also look at the Venn diagram to see the amount of shared variance between the variables. It is also possible drag the data points to see how the correlation is influenced by outliers.

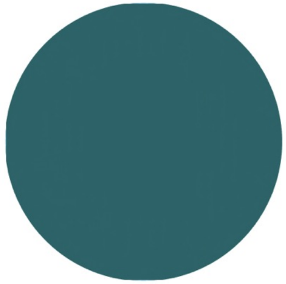
Slide me




Correlation: **1**

Sample size 100 New sample

Shared variance: **100%**



Posters

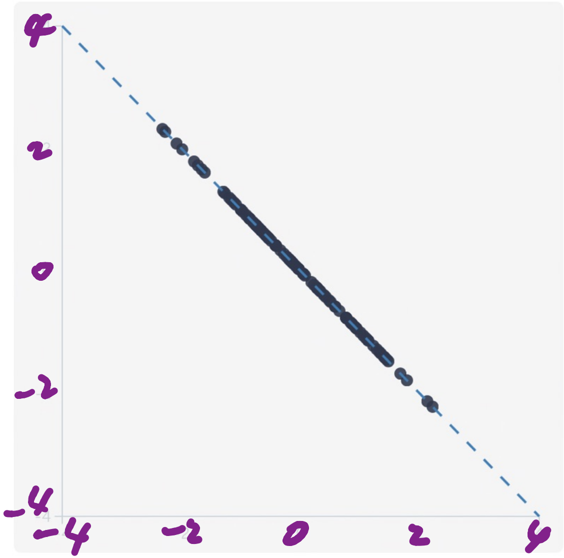
8:40 AM Tue Sep 1

rpsychologist.com

95%

Correlation is one of the most widely used tools in statistics. The correlation coefficient summarizes the association between two variables. In this visualization I show a scatter plot of two variables with a given correlation. The variables are samples from the standard normal distribution, which are then transformed to have a given correlation by using Cholesky decomposition. By moving the slider you will see how the shape of the data changes as the association becomes stronger or weaker. You can also look at the Venn diagram to see the amount of shared variance between the variables. It is also possible drag the data points to see how the correlation is influenced by outliers.

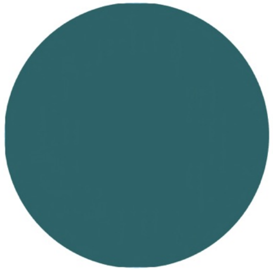
Slide me




## Correlation: -1

Sample size 100 New sample

## Shared variance: **100%**



Posters

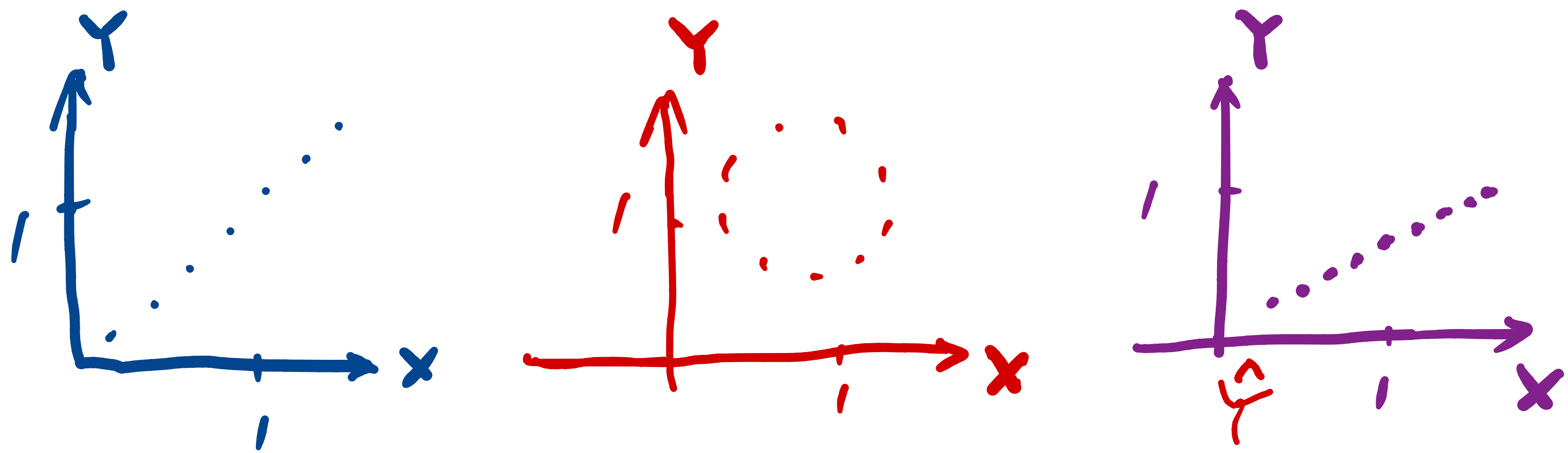
# The Properties of Correlation Coefficient

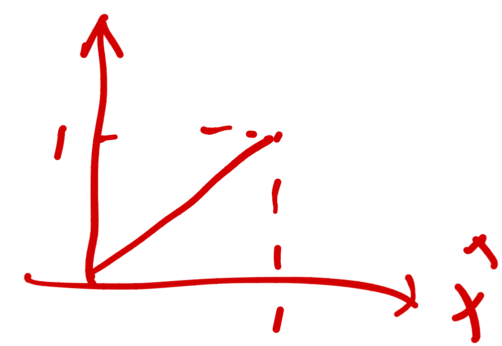
- The correlation coefficient is bounded within [-1, 1]

$corr(\{(x_i, y_i)\}) = 1$ if and only if $\widehat{x_i} = \widehat{y_i}$

$corr(\{(x_i, y_i)\}) = -1$ if and only if $\widehat{x_i} = -\widehat{y_i}$

# Which of the following has correlation coefficient equal to 1?




A. Left and right

B. Left

C. Middle

# Concept of Correlation Coefficient's bound

- The correlation coefficient can be written as

$$corr(\{(x_i, y_i)\}) = \frac{1}{N}\sum_{i=1}^{N} \widehat{x_i}\widehat{y_i}$$

$\vec{u} \cdot \vec{v}$

$= \sum_{i=1}^{N} u_i \cdot v_i$

$$\boxed{corr(\{(x_i, y_i)\}) = \sum_{i=1}^{N} \frac{\widehat{x_i}}{\sqrt{N}}\frac{\widehat{y_i}}{\sqrt{N}}}$$

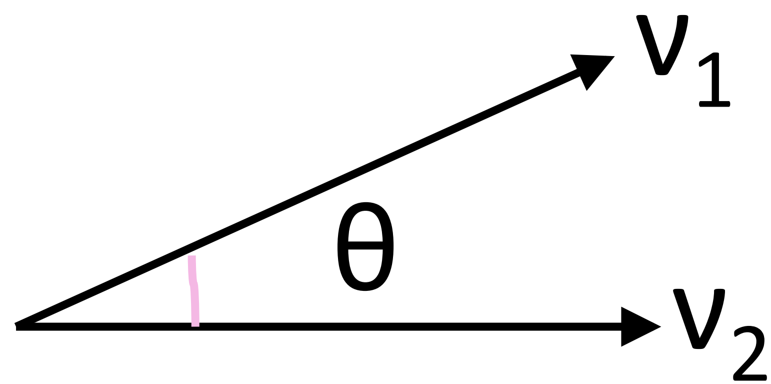
- It's the inner product of two vectors

$$\left\langle \frac{\widehat{x_1}}{\sqrt{N}}, \quad \ldots \quad \frac{\widehat{x_N}}{\sqrt{N}} \right\rangle \text{ and } \left\langle \frac{\widehat{y_1}}{\sqrt{N}}, \quad \ldots \quad \frac{\widehat{y_N}}{\sqrt{N}} \right\rangle$$

# Inner product

- Inner product's geometric meaning:

$$\underbrace{|\nu_1|}_{=1}\,\underbrace{|\nu_2|}_{=1}\,cos(\theta)$$



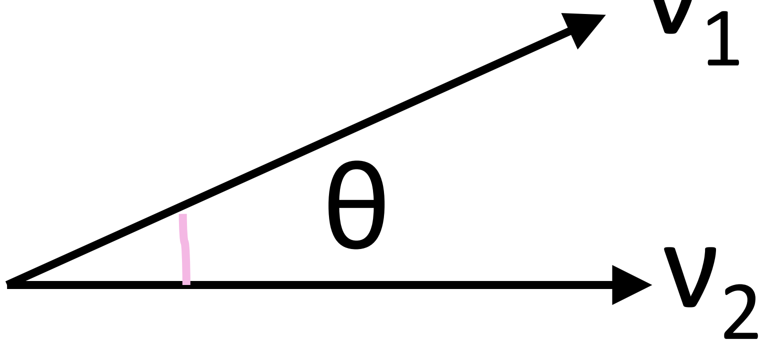

- Lengths of both vectors

$$v_1 = \left\langle \frac{\widehat{x_1}}{\sqrt{N}}, \quad \cdots \quad \frac{\widehat{x_N}}{\sqrt{N}} \right\rangle \qquad v_2 = \left\langle \frac{\widehat{y_1}}{\sqrt{N}}, \quad \cdots \quad \frac{\widehat{y_N}}{\sqrt{N}} \right\rangle$$

are 1

# Bound of correlation coefficient

$$|corr(\{(x_i, y_i)\})| = |cos(\theta)| \leq 1$$



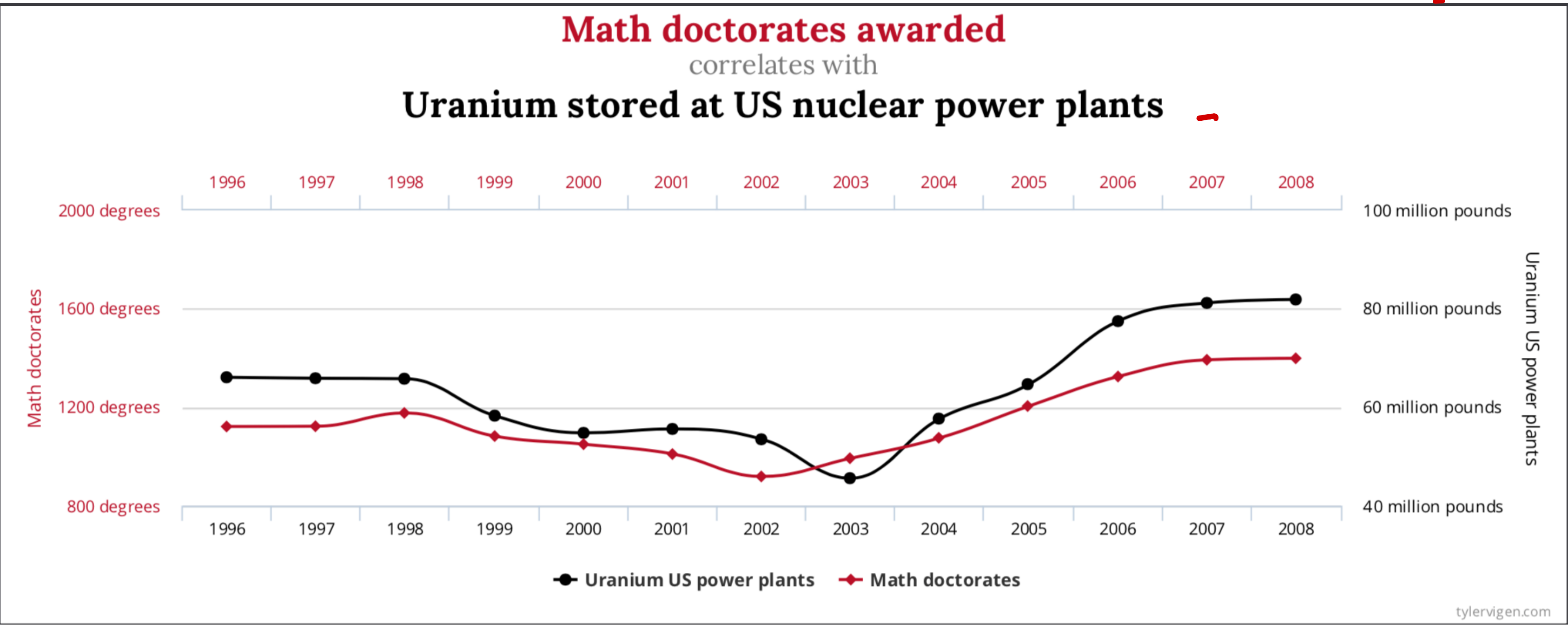

$$v_1 = \left\langle \frac{\widehat{x_1}}{\sqrt{N}}, \quad \cdots \quad \frac{\widehat{x_N}}{\sqrt{N}} \right\rangle \quad v_2 = \left\langle \frac{\widehat{y_1}}{\sqrt{N}}, \quad \cdots \quad \frac{\widehat{y_N}}{\sqrt{N}} \right\rangle$$

# The Properties of Correlation Coefficient

- Symmetric
- Translating invariant
- Scaling only may change sign
- bounded within [-1, 1]

# Using correlation to predict

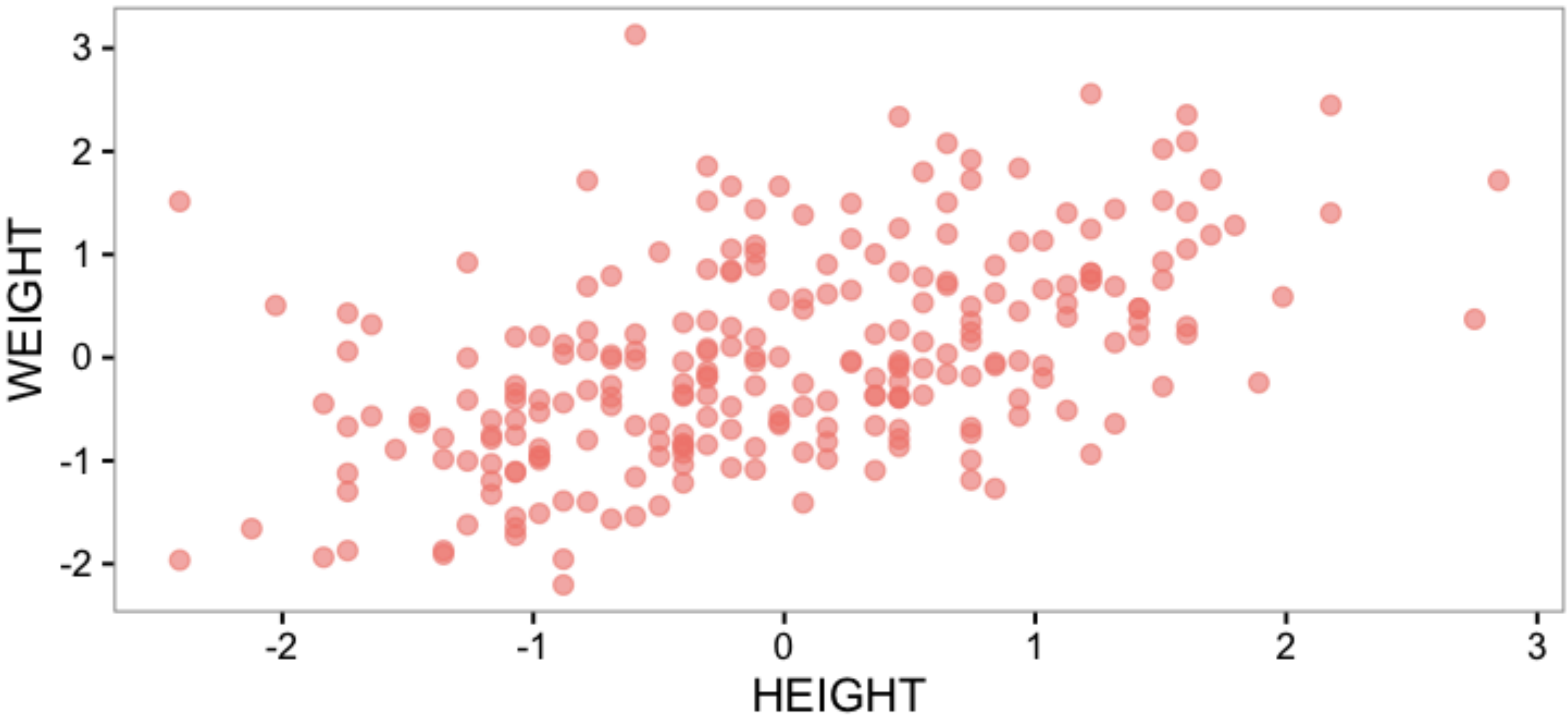
- **Caution**! Correlation is **NOT** Causation




Credit: Tyler Vigen

# How do we go about the prediction?

- Removed of outliers & standardized



# Using correlation to predict

- Given a correlated data set $\{(x_i, y_i)\}$

  we can predict a value ${y_0}^p$ that goes with a value $x_0$

- In standard coordinates $\{(\widehat{x_i}, \widehat{y_i})\}$

  we can predict a value $\widehat{y_0}^p$ that goes with a value $\widehat{x_0}$

# Q:

- Which coordinates will you use for the predictor using correlation?

  A. Standard coordinates

  B. Original coordinates

  C. Either

# Linear predictor and its error

- We will assume that our predictor is linear

$$\widehat{y}^{\,p} = a\,\widehat{x} + b$$

- We denote the prediction at each $\widehat{x_i}$ in the data set as $\widehat{y_i}^p$

$$\widehat{y_i}^{\,p} = a\,\widehat{x_i} + b$$

- The error in the prediction is denoted $u_i$

$$u_i = \widehat{y_i} - \widehat{y_i}^{\,p} = \widehat{y_i} - a\,\widehat{x_i} - b$$

# Require the mean of error to be zero

We would try to make the mean of error equal to zero so that it is also centered around 0 as the standardized data:

centered

$$
\begin{aligned}
\underline{\mathrm{mean}(\{u_i\})} &= \mathrm{mean}(\{\hat{y} - \hat{y}^p\}) \\
&= \mathrm{mean}(\{\hat{y} - a\hat{x} - b\}) \\
&= \underbrace{\mathrm{mean}(\{\hat{y}\})}_{0} - a \cdot \underbrace{\mathrm{mean}(\{\hat{x}\})}_{0} - b \\
&= -b = 0 \\
&\Rightarrow b = 0
\end{aligned}
$$

# Require the variance of error is minimal

minimize $\sigma^2$

$$\sigma^2 = \text{mean}(\{(u_i - \underbrace{\text{mean}(\{u_i\})}_{\to 0})^2\})$$

$$= \text{mean}(\{u_i\}^2)$$

$$= \text{mean}(\{(\hat{y} - \hat{y}^p)^2\})$$

$$= \text{mean}(\{(\hat{y} - a\hat{x} - \cancel{b})^2\})$$

$$= \text{mean}(\{\hat{y}^2 - 2a\hat{x}\hat{y} + a^2\hat{x}^2\})$$

$$= \text{mean}(\{\hat{y}^2\}) - 2a\,\underbrace{\text{mean}(\{\hat{x}\hat{y}\})}_{\text{corr}} + a^2\,\underline{\text{mean}(\{\hat{x}^2\})}$$

$$\text{mean}(\{\hat{y}^2\})$$

$$= \text{mean}(\{(\hat{y} - 0)^2\})$$

$$= \text{var}(\hat{y}) = 1$$

$$= 1 - 2ar + a^2$$

$$\frac{d(1 - 2ar + a^2)}{da} = 0$$

$$-2r + 2a = 0$$

# Require the variance of error is minimal

$\boxed{a = r}$

# Here is the linear predictor!

$$\widehat{y}^{\,p} = r\ \widehat{x}$$

$$\hat{y}^p = a\hat{x} + b$$

$$a = r$$

$$b = 0$$

Correlation coefficient

# Prediction Formula

- In standard coordinates

$$\widehat{y_0}^{\,p} = r\ \widehat{x_0} \quad \text{where} \quad r = corr(\{(x_i, y_i)\})$$

- In original coordinates

$$\frac{y_0^p - mean(\{y_i\})}{std(\{y_i\})} = r\frac{x_0 - mean(\{x_i\})}{std(\{x_i\})}$$

# Root-mean-square (RMS) prediction error

- $$Given \qquad var(\{u_i\}) = \underline{1 - 2ar + a^2}$$
  $$\& \qquad a = r$$

  $$var(\{u_i\}) = 1 - r^2$$

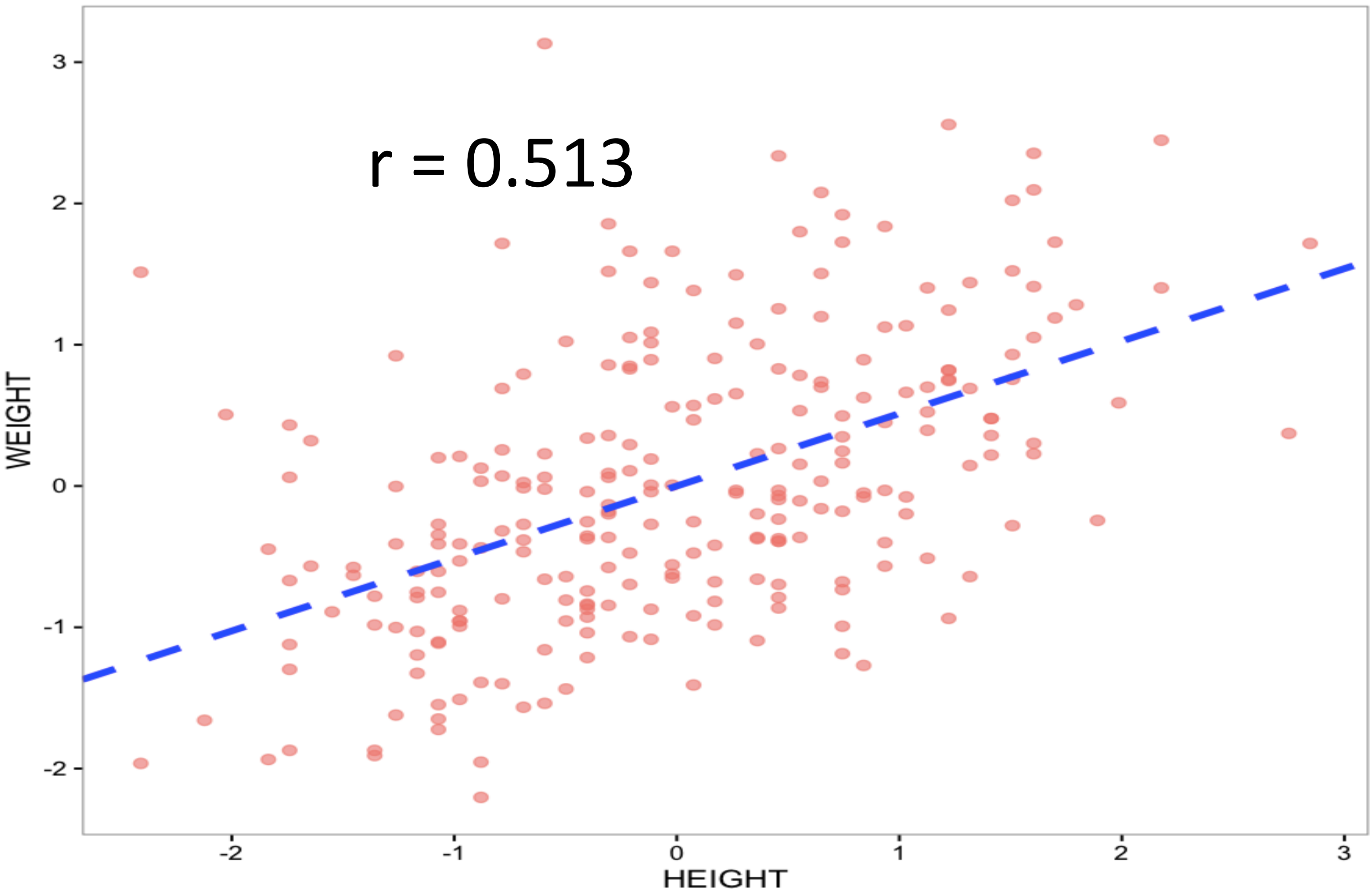
  if $r=1$ $\quad var(\{u_i\})=0$

- $$\boxed{\begin{aligned} RMS \;\; error &= \sqrt{mean(\{u_i^2\})} \\ &= \sqrt{var(\{u_i\})} \\ &= \sqrt{1 - r^2} \end{aligned}}$$

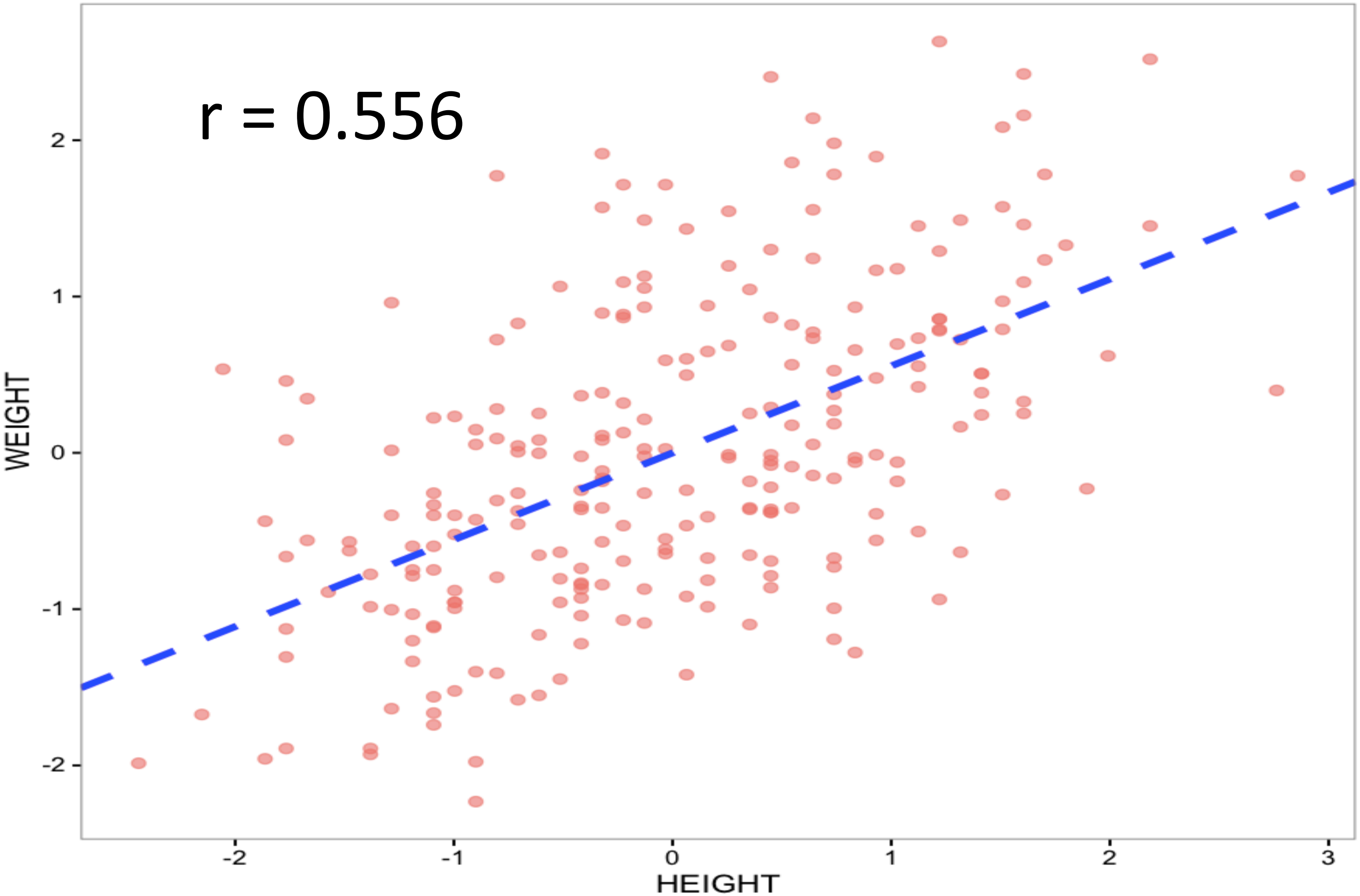
# See the error through simulation

https://rpsychologist.com/d3/correlation/

# Example: Body Fat data



# Example: remove 2 more outliers



# Heatmap

- Display matrix of data via gradient of color(s)

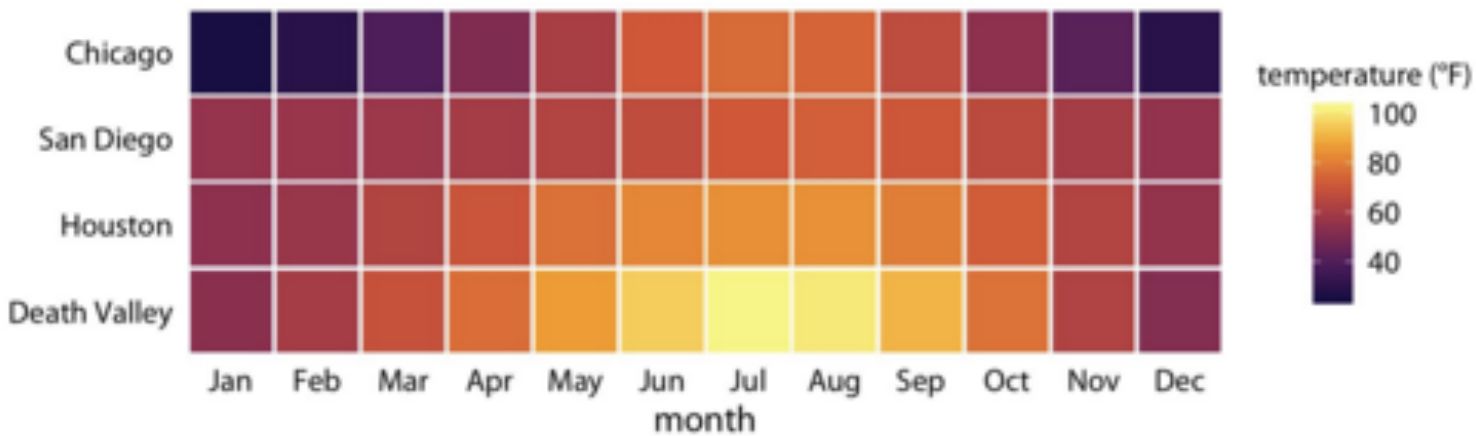
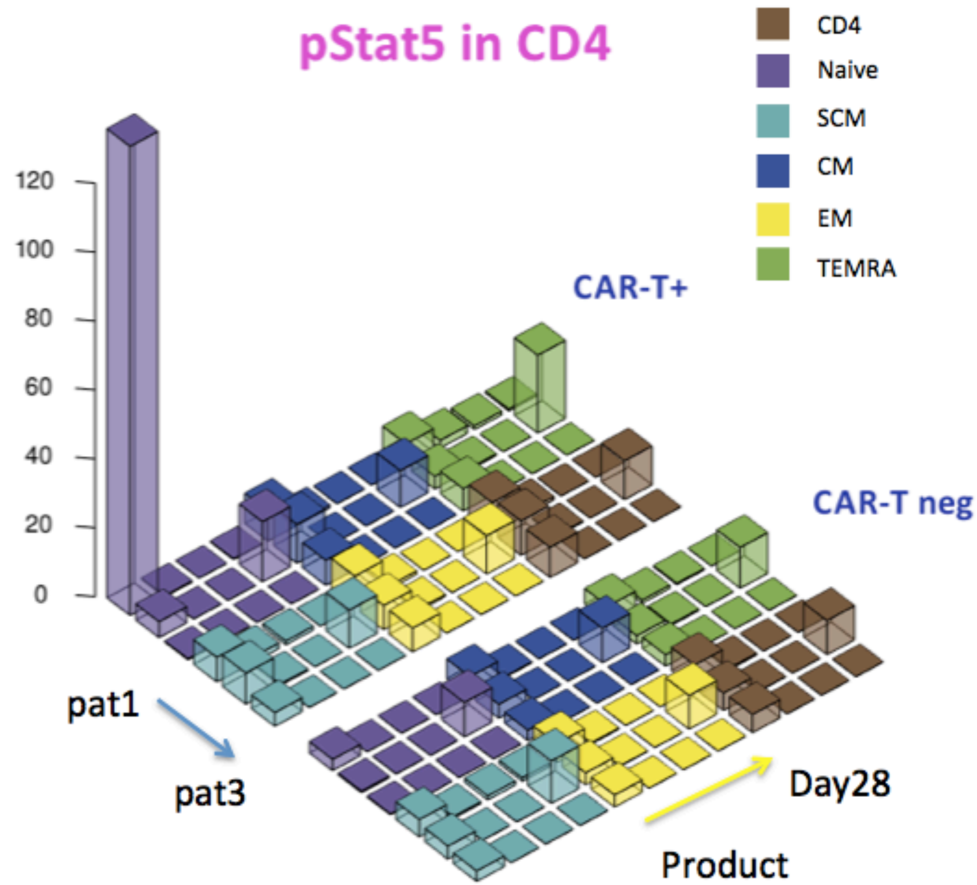



***Figure 2-4. Monthly normal mean temperatures for four locations in the US. Data source: NOAA.***

Summarization of 4 locations' annual mean temperature by month

# 3D bar chart

- Transparent 3D bar chart is good for small # of samples across categories



# Relationship between data feature and time

- Example: How does Amazon's stock change over 1 years?

  take out the pair of

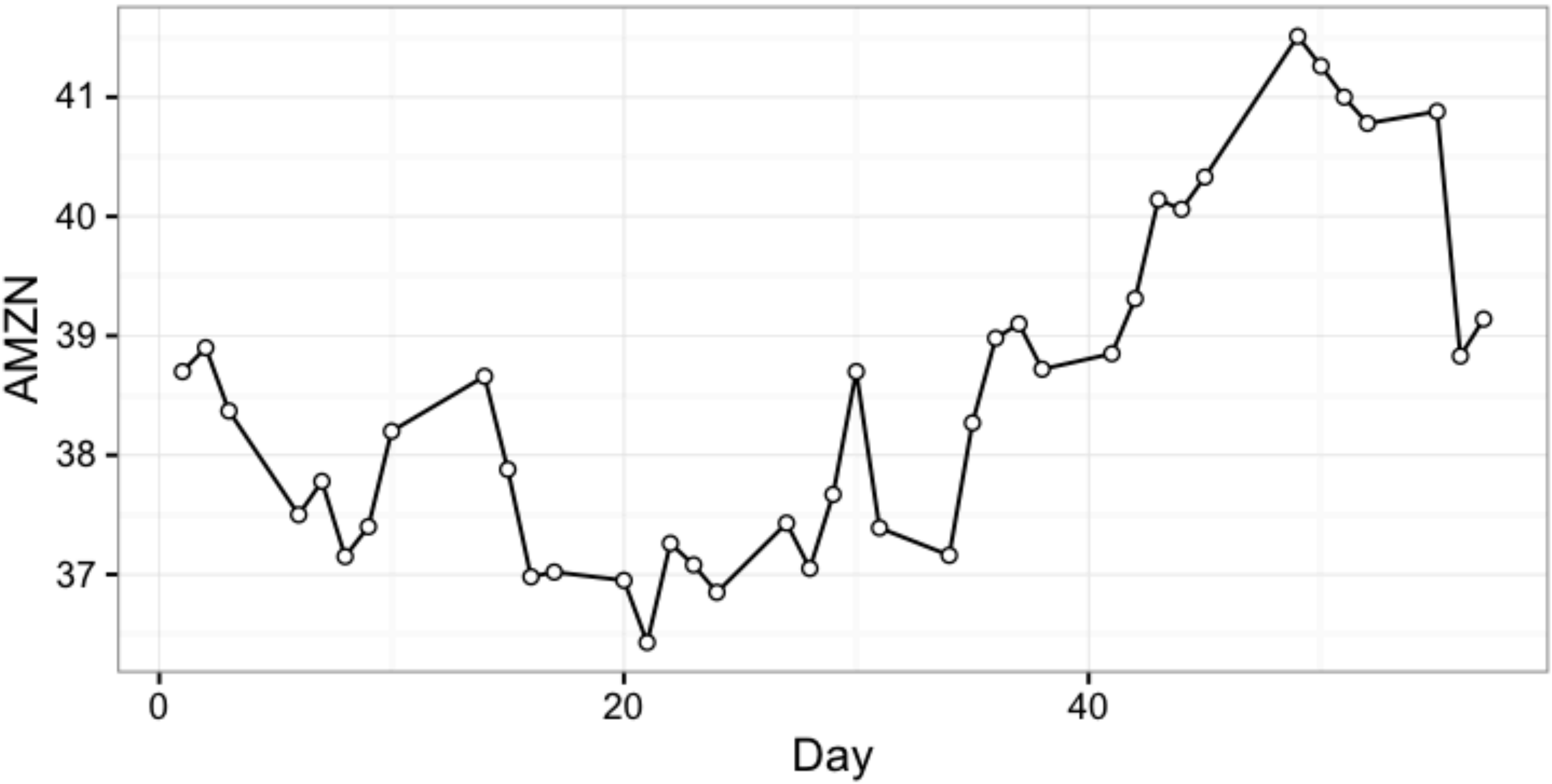
  features

  x: Day

  y: AMZN

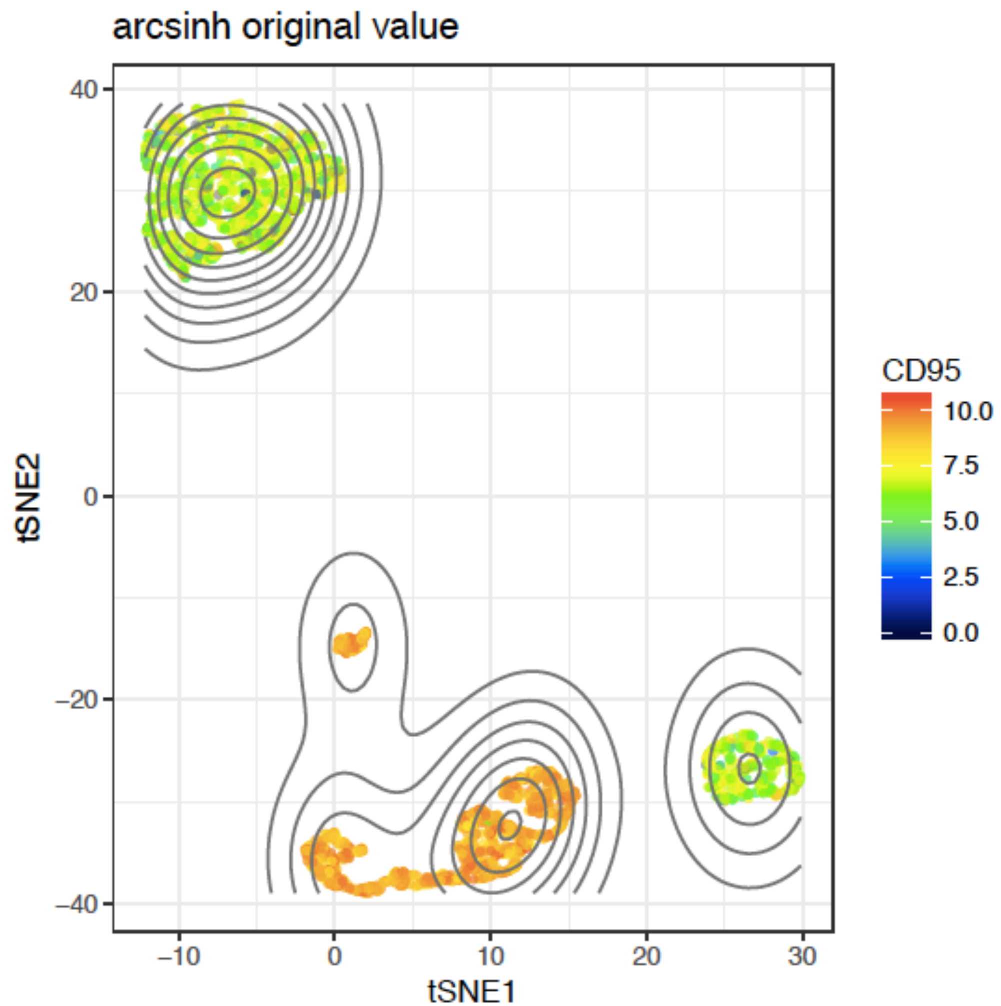
| Day | AMZN | DUK | KO |
|---|---|---|---|
| 1 | 38.700001 | 34.971017 | 17.874906 |
| 2 | 38.900002 | 35.044103 | 17.882263 |
| 3 | 38.369999 | 34.240172 | 17.757161 |
| 6 | 37.5 | 34.294985 | 17.871225 |
| 7 | 37.779999 | 34.130544 | 17.885944 |
| 8 | 37.150002 | 33.984374 | 17.9117 |
| 9 | 37.400002 | 34.075731 | 17.933777 |
| 10 | 38.200001 | 33.91129 | 17.863866 |
| 14 | 38.66 | 34.020917 | 17.845469 |
| 15 | 37.880001 | 33.966104 | 17.882263 |
| 16 | 36.98 | 34.130544 | 17.790276 |
| 17 | 37.02 | 34.240172 | 17.757161 |
| 20 | 36.950001 | 34.057458 | 17.672533 |
| 21 | 36.43 | 34.112272 | 17.705649 |
| 22 | 37.259998 | 34.258442 | 17.709329 |
| 23 | 37.080002 | 34.569051 | 17.639418 |
| 24 | 36.849998 | 34.861392 | 17.598945 |

# Time Series Plot: Stock of Amazon



# Scatter plot

- Coupled with heatmap to show a 3rd feature



# Assignments

- Finish reading Chapter 2 of the textbook
- Next time: Probability a first look

# Additional References


- Charles M. Grinstead and J. Laurie Snell "Introduction to Probability"
- Morris H. Degroot and Mark J. Schervish "Probability and Statistics"

# See you next time

*See You!*