Recap

• (Ch 1-2) Looking at data and relationships
• (Ch 3-5) Probability

Today and the next few lectures

• (Ch 6-9) Statistical inference
  • (Ch 6) How to draw general conclusions from a sample of the population
  • (Ch 7) How to assess the significance of the evidence against a hypothesis
  • (Ch 9) How to infer a probability model from a dataset
Motivation: midterm grading example

• In about a week, your instructor is going to grade 100 midterms

• Being impatient to know how the class did, he will first grade a sample of 5 randomly selected exams
  • Suppose he gives the following scores \{120, 130, 140, 140, 150\}
  • So the realized sample mean is 136

• What does he now know about the population mean?
  • What is his best guess for the population mean? 136
  • How confident should he be in his best guess?

Would he be more confident if he had graded say 10 exams
Motivation: election polling example

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Source: fivethirtyeight.com

- The Southern Illinois University poll tells us:
  - The sample consists of 715 likely voters
  - Pritzker’s vote percentage has realized sample mean equal to 49%

- What do we know about the population mean of Pritzker’s Sep 24-29 vote percentage, where the population is all likely voters in Illinois?
Population

• The population is the entire dataset \( \{x\} \)
  • The population size is \( N_p \)
  • The population mean is \( \text{popmean}(\{x\}) \) and is a number (not a random variable)
  • The population standard deviation is \( \text{popsd}(\{x\}) \) and is a number

• We have formulas for \( \text{popmean}(\{x\}) \) and \( \text{popsd}(\{x\}) \) from Lec 1
Sample

- The sample is a **random** subset of the population \( \{x\} \), where the sampling is done with replacement
  - The sample size is \( N \), assumed to be much less than population size \( N_p \)
  - The sample mean is \( X^{(N)} \) and is a **random variable**

- Coming up, we will obtain expressions for
  - The expected value of the sample mean \( E[X^{(N)}] \)
  - The standard deviation of the sample mean \( \text{std}[X^{(N)}] \)
    (also known as **standard error**)

Why are we doing this?

A narrower distribution (i.e., smaller $\text{std}(x^{(w)})$) would give us more confidence in the estimate of $\text{popmean}(\{x\})$. $x^{(w)}$ realized as $\text{mean}(\{x\})$ is our estimate of $\text{popmean}(\{x\})$. 

The distribution of $x^{(w)}$
Expected value of the sample mean ...

• The sample mean is the average of IID samples

\[ X^{(N)} = \frac{1}{N} (X_1 + X_2 + \cdots + X_N) \]

• By linearity of expectation and the fact that the sample mean \( X^{(1)} \) of 1 sample is the sample itself

\[ E[X^{(N)}] = \frac{1}{N} (E[X^{(1)}] + E[X^{(1)}] + \cdots + E[X^{(1)}]) = E[X^{(1)}] \]
... is the population mean

- Since each sample is sampled uniformly from the population
  \[ E[X^{(1)}] = \text{popmean}(\{x\}) \]

- Therefore
  \[ E[X^{(N)}] = \text{popmean}(\{x\}) \]

- We say that \( X^{(N)} \) is an unbiased estimator for \( \text{popmean}(\{x\}) \)

- We actually proved something very similar in Lec 9 in the proof of WLLN using slightly different notation
What does this mean?

• The sample mean is an unbiased estimate of the population mean as long as the samples have been drawn independently and with equal probability from the population

• Examples of poor sampling
  • To estimate the average height of people in Champaign, should I sample at the local daycare or perhaps try at the basketball team practice?
  • To poll likely voters, should we call them only at landline phone numbers? But it is costly for pollsters to call cellphones since robocalling them is illegal
Standard deviation of the sample mean

- We can also rewrite another result from Lec 9 as

\[ \text{var}[X^{(N)}] = \frac{\text{popvar}({x})}{N} \]

- So the **standard error** (standard deviation of the sample mean) is

\[ \text{stderr}({x}) = \text{std}[X^{(N)}] = \frac{\text{popsd}({x})}{\sqrt{N}} \]

- But we don’t know the value of the population standard deviation
Estimating the population standard deviation

• We will try to use the realized sample to estimate \( \text{popsd}([x]) \)

• Does this work?

\[
\sqrt{\frac{1}{N} \sum_{x_i \in \text{sample}} (x_i - \text{mean}([x_i]))^2}
\]

• No! This formula turns out to underestimate \( \text{popsd}([x]) \) on average

• To make it an unbiased estimator, we must multiply it by \( \sqrt{\frac{N}{N-1}} \)
Unbiased estimate of population std. dev.

- The unbiased estimate of \( \text{popsd}(\{x\}) \) is defined as

\[
\text{std\textunderscore unbiased}(\{x\}) = \sqrt{\frac{1}{N-1} \sum_{x_i \in \text{sample}} (x_i - \text{mean}(\{x_i\}))^2}
\]

- So the standard error is estimated as

\[
\text{stderr}(\{x\}) = \text{std}[X^{(N)}] = \frac{\text{popsd}(\{x\})}{\sqrt{N}} = \frac{\text{std\textunderscore unbiased}(\{x\})}{\sqrt{N}}
\]
Standard error: midterm grading example

The realized sample of scores is \{120, 130, 140, 140, 150\} with \( N = 5 \)

\[
\text{mean}(\{120, 130, 140, 140, 150\}) = 136
\]

\[
\text{stdunbiased}(\{x\}) = \sqrt{\frac{1}{5-1} \left( (120-136)^2 + (130-136)^2 + 2(140-136)^2 + (150-136)^2 \right) \approx 11.4
\]

\[
\text{stderr}(\{x\}) = \frac{11.4}{\sqrt{5}} \approx 5.1
\]

So we estimate the population mean as 136 with standard error 5.1
Standard error: election polling example

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Number of sampled voters who selected Pritzker

\[ 715(0.49) \approx 350 \]

Number of sampled voters who did not select Pritzker

\[ 715(0.51) = 365 \]

Source: fivethirtyeight.com
Standard error: election polling example

\[
\text{stdunbiased}\{x\} = \sqrt{\frac{1}{715-1} \left( 350 \left(1 - 0.49\right)^2 + 365 \left(0 - 0.49\right)^2 \right)} \approx 0.50
\]

\[
\text{stderr}\{x\} = \frac{0.5}{\sqrt{715}} \approx 0.019
\]

So we estimate the population mean as 49% with standard error 1.9%
Interpreting the standard error when $N \geq 30$

If the sample size $N \geq 30$, the Central Limit Theorem tells us that we can approximate the distribution of the sample mean $X^{(N)}$ as a normal distribution with $\mu = \text{popmean}(\{x\})$ and $\sigma = \text{stderr}(\{x\})$. 

- 99.7% of the data are within 3 standard deviations of the mean.
- 95% within 2 standard deviations.
- 68% within 1 standard deviation.

$\mu - 3\sigma \quad \mu - 2\sigma \quad \mu - \sigma \quad \mu \quad \mu + \sigma \quad \mu + 2\sigma \quad \mu + 3\sigma$
Confidence intervals when $N \geq 30$

• For about 68% of samples

$$\text{mean}(\{x\}) - \text{stderr}(\{x\}) \leq \text{popmean}(\{x\}) \leq \text{mean}(\{x\}) + \text{stderr}(\{x\})$$

• For about 95% of samples

$$\text{mean}(\{x\}) - (2)\text{stderr}(\{x\}) \leq \text{popmean}(\{x\}) \leq \text{mean}(\{x\}) + (2)\text{stderr}(\{x\})$$

• For about 99% of samples

$$\text{mean}(\{x\}) - (3)\text{stderr}(\{x\}) \leq \text{popmean}(\{x\}) \leq \text{mean}(\{x\}) + (3)\text{stderr}(\{x\})$$
Confidence interval: election polling example

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\[ n = 715 \gg 30 \]

- We estimated the population mean as 49% with standard error 1.9%
- The 99% confidence interval for Pritzker’s vote percentage is

\[
[49\% - (3)1.9\%, 49\% + (3)1.9\%] = [43.3\%, 54.7\%]
\]

Source: fivethirtyeight.com
Confidence intervals when $N < 30$

• If the sample size $N < 30$, we should not use the normal distribution to create confidence intervals.

• Instead we use Student’s t-distribution with its parameter (called degrees of freedom) set to $N - 1$.

• In the midterm grading example, we should use Student’s t-distribution with degrees of freedom set to 4 since $N = 5$. 