Recap

- (Ch 4) Random variables
  - Expected value, variance and covariance
  - Towards the weak law of large numbers

Today

- (Ch 4) Random variables
  - Weak law of large numbers
  - Simulation examples
  - Continuous random variables

Towards the weak law of large numbers

- The weak law says that if we repeat an experiment many times, the average of the observations will “converge” to the expected value
- The weak law justifies using simulations (instead of calculations) to estimate the expected values of random variables

Markov's inequality

- For any random variable X and constant $a > 0$

$$P(|X| \geq a) \leq \frac{E[|X|]}{a}$$

- In words, a random variable is unlikely to have an absolute value much larger than the mean of its absolute value

- For example, if $a = 10E[|X|]$

$$P(|X| \geq 10E[|X|]) \leq 0.1$$

Chebyshev's inequality

- For any random variable X and constant $a > 0$

$$P(|X - E[X]| \geq a) \leq \frac{\mathrm{var}[X]}{a^2}$$

- To rephrase, let $a = k\sigma$ where $\sigma = \mathrm{std}[X]$

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

- In words, the probability that X is greater than k standard deviations from the mean is small

Proof of Chebyshev's inequality

- Apply Markov's inequality to $U = (X - E[X])^2$, so $E[U] = \text{var}[X]$

$$P(|U| \geq w) \leq \frac{E[|U|]}{w} = \frac{E[U]}{w} = \frac{\text{var}[X]}{w}$$

- Substitute $U = (X - E[X])^2$ and $w = a^2$

$$\underbrace{P((X - E[X])^2 \geq a^2)} \leq \frac{\text{var}[X]}{a^2}$$

$$P(|X - E[X]| \geq a)$$

IID samples and the sample mean

- Say you have a random variable X with probability distribution $P(X)$

randomly

- Say you generate independent samples $\{x_i\}$ so that their histogram resembles $P(X)$ more closely as the number of samples increases

- We call $\{x_i\}$ independent identically distributed **(IID) samples** of $P(X)$

- The **sample mean** of $\{x_i\}$ is a random variable: $X_N = \frac{1}{N}\sum_{i=1}^{N} x_i$

Expected value of sample mean

- By linearity of expected value

$$E[X_N] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i]$$

- Since each x_i is a sample drawn from $P(X)$, we have $E[x_i] = E[X]$

$$E[X_N] = \frac{1}{N}\sum_{i=1}^{N} E[X] = E[X]$$

Variance of sample mean

- By the scaling property of variance and independence of samples x_i

$$\operatorname{var}[X_N] = \operatorname{var}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N^2}\operatorname{var}\left[\sum_{i=1}^{N} x_i\right] = \frac{1}{N^2}\sum_{i=1}^{N}\operatorname{var}[x_i]$$

- Since each x_i is drawn from $P(X)$, we have $\operatorname{var}[x_i] = \operatorname{var}[X]$

$$\operatorname{var}[X_N] = \frac{1}{N^2}\sum_{i=1}^{N}\operatorname{var}[X] = \frac{\operatorname{var}[X]}{N}$$

Weak law of large numbers (WLLN)

- Given a random variable X with finite variance, probability distribution $P(X)$ and sample mean X_N

- For any positive number ϵ

$$\lim_{N \to \infty} P(|X_N - E[X]| \geq \epsilon) = 0$$

- In words: for a large enough set of IID samples, the sample mean X_N will be very close to the expected value $E[X]$ with high probability

Proof of WLLN

- Apply Chebyshev's inequality to the sample mean X_N

$$P(|X_N - E[X_N]| \geq \epsilon) \leq \frac{var[X_N]}{\epsilon^2}$$

- Substitute $E[X_N] = E[X]$ and substitute $\text{var}[X_N] = \text{var}[X]/N$

$$P(|X_N - E[X]| \geq \epsilon) \leq \frac{var[X]}{N\epsilon^2} \xrightarrow[N \to \infty]{} 0$$

Simulation: airline overbooking example

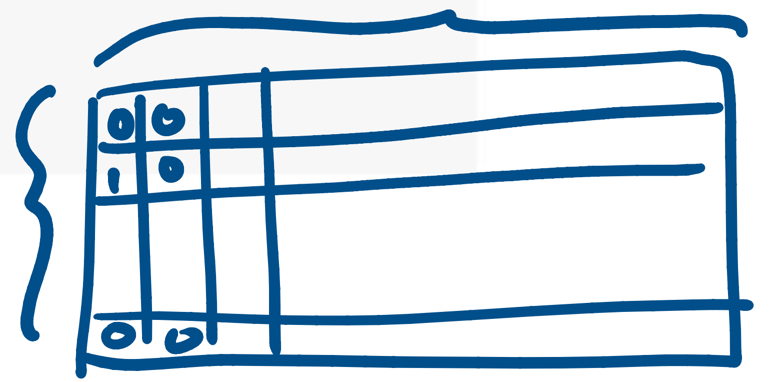
An airline has a flight with 6 seats. They always sell 12 tickets for this flight and ticket holders show up independently with probability p. Plot the following quantities as a function of p.

- Expected number of ticket holders that show up
- Probability that the flight is overbooked
- Expected number of ticket holders who show up but don’t fly given that the flight is overbooked

Approximating expected value

Expected number of ticket holders that show up

```
results = np.zeros((10, 2))
numTrials = 100000
numTickets = 12
numSeats = 6
for i,p in enumerate(np.linspace(0.1, 1.0, num=10)):
    arrivals = np.random.random((numTickets, numTrials)) < p
    numArrivals = arrivals.sum(axis=0)
    results[i] = [p, numArrivals.mean()]
results
```



Approximating probability

Probability that the flight is overbooked

```
results = np.zeros((10, 2))
numTrials = 100000
numTickets = 12
numSeats = 6
for i,p in enumerate(np.linspace(0.1, 1.0, num=10)):
    arrivals = np.random.random((numTickets, numTrials)) < p
    numArrivals = arrivals.sum(axis=0)
    indicatorOverbooked = numArrivals > numSeats
    results[i] = [p, indicatorOverbooked.mean()]
results
```

Approximating conditional expected value

Expected number of ticket holders who show up but don't fly given that the flight is overbooked

```
results = np.zeros((10, 2))
numTrials = 100000
numTickets = 12
numSeats = 6
for i,p in enumerate(np.linspace(0.1, 1.0, num=10)):
    arrivals = np.random.random((numTickets, numTrials)) < p
    numArrivals = arrivals.sum(axis=0)
    indicatorOverbooked = numArrivals > numSeats
    numDontFly = numArrivals[indicatorOverbooked] - numSeats
    results[i] = [p, numDontFly.mean()]
results
```

Continuous random variables

- So far we have been talking about discrete random variables

- Some random variables can take on a continuous set of values
  - Temperature
  - Height
  - Sample mean X_N (whoops!)

- Defining samples spaces, outcomes and events for continuous random variables is beyond the scope of CS 361

Probability density function (pdf)

- For a continuous random variable X, the probability that $X = x$ is essentially zero for all (or most) x, so we can't define $P(X = x)$

- Instead, we define the **probability density function (pdf)** over an infinitesimally small interval dx

$$p(x)dx = P(X \in [x, x + dx])$$

- For $a < b$

$$\int_a^b p(x)\,dx = P(X \in [a, b])$$

Properties of the probability density function

- $p(x)$ is a bit like a discrete random variable's probability distribution
  - $p(x) \geq 0$ for all x
  - The probability of X taking some value is 1

$$\int_{-\infty}^{\infty} p(x)\, dx = 1$$

- $p(x)$ is **not** like a discrete random variable's probability distribution
  - $p(x)$ is not the probability that $X = x$
  - $p(x)$ can exceed 1

Probability density function: height example

- Suppose we heard that Napoleon was 62.5 inches tall, rounded up to the nearest half inch. What is the pdf of his height H?

- Assume that H is equally likely to be any value in $(62, 62.5]$ inches

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$$p(h) = \begin{cases} c & \text{if } h \in (62, 62.5] \\ 0 & \text{if } h \notin (62, 62.5] \end{cases}$$ where c is a constant

- Then

$$1 = \int_{-\infty}^{\infty} p(h)\, dh = \int_{62}^{62.5} c\, dh = \frac{c}{2} \Longrightarrow c = 2$$