Recap

• (Ch 4) Random variables
  • Probability distribution
  • Joint probability distribution
  • Conditional probability distribution

Today

• (Ch 4) Random variables
  • Expected value, variance and covariance
  • Towards the weak law of large numbers
Expected value

• The **expected value** of a random variable $X$ is

$$E[X] = \sum_x xP(x)$$

• The expected value is a weighted average of the values taken by $X$
Expected value: gambling example

• Let’s bet on a coin toss
  • It comes up heads with probability $p$ and tails with probability $1 - p$
  • If it comes up heads I pay you $10; otherwise, you pay me $10

• For what values of $p$ is this a good game for you?
Expected value as mean

• Suppose we have a data set \( \{x_i\} \) of \( N \) data points. Let’s build an empirical probability distribution from the data set by assigning each data point with probability \( \frac{1}{N} \).

\[
E[X] = \sum_i x_i P(x_i) = \frac{1}{N} \sum_i x_i = \text{mean}(\{x_i\})
\]

• The expected value is also called the mean
Linearity properties of expected value

• For random variables $X$ and $Y$ and constant $k$

\[
E[X + Y] = E[X] + E[Y]
\]

\[
E[kX] = kE[X]
\]

• These properties follow from interpreting expected values as means of data sets
Expected value of a function of $X$

- If $f$ is a function of a random variable $X$, then $Y = f(X)$ is a random variable too.

- The **expected value** of $Y = f(X)$ is

$$E[Y] = E[f(X)] = \sum_x f(x)P(x)$$
Expected value: online gambling example

Let’s make the same bet as before, but now you pay a 10% fee on all transactions. For what values of $p$ is this a good game for you?

$X$ is your net winnings before fees. We already know that $E[X] = 20p - 10$
Variance and standard deviation

• The **variance** of a random variable $X$ is

$$\text{var}[X] = E[(X - E[X])^2]$$

• The **standard deviation** of a random variable $X$ is

$$\text{std}[X] = \sqrt{\text{var}[X]}$$
Properties of variance

• For random variables $X$ and $Y$ and constant $k$

\[ \text{var}[k] = 0 \]

\[ \text{var}[X] \geq 0 \]

\[ \text{var}[kX] = k^2 \text{var}[X] \]

• If $X$ and $Y$ are independent

\[ \text{var}[X + Y] = \text{var}[X] + \text{var}[Y] \]
A neater expression for variance

\[ \text{var}[X] = E[(X - \mu)^2] \]  \quad \text{where } \mu = E[X]
Variance: online gambling example

Let’s make the same bet as before. What is the variance of your net winnings before fees?

$X$ is your net winnings before fees. We already know that $E[X] = 20p - 10$
Covariance

• The **covariance** of random variables $X$ and $Y$ is

\[
\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])]
\]

• Note that

\[
\text{cov}(X, X) = E[(X - E[X])^2] = \text{var}[X]
\]
Properties of covariance

• A neater expression for covariance (similar derivation as for variance)

\[
cov(X, Y) = E[XY] - E[X]E[Y]
\]

• If \(X\) and \(Y\) are independent, the following are proven in the book

\[
E[XY] = E[X]E[Y]
\]

\[
cov(X, Y) = 0
\]
Covariance: online gambling example

Let’s make the same bet as before. What is the covariance of your net winnings before fees and your net winnings after fees?

\( X \) and \( Y \) are your net winnings before and after fees. So, \( E[X] = 20p - 10 \) and \( E[Y] = 20p - 11 \).
Towards the weak law of large numbers

• The weak law says that if we repeat an experiment many times, the average of the observations will “converge” to the expected value

• For example, if you actually repeat the bet discussed in this lecture, your average winnings after fees will “converge” to $E[Y] = 20p - 11$

• The weak law justifies using simulations (instead of calculations) to estimate the expected values of random variables
Indicator functions

• An indicator function for an event $E$ is a function of $X$ such that

• The expected value of the indicator function is the probability of $E$
Markov’s inequality

• For any random variable $X$ and constant $a > 0$

$$P(|X| \geq a) \leq \frac{E[|X|]}{a}$$

• In words, a random variable is unlikely to have an absolute value much larger than the mean of its absolute value

• For example, if $a = 10E[|X|]$

$$P(|X| \geq 10E[|X|]) \leq 0.1$$
Proof of Markov’s inequality
Chebyshev’s inequality

• For any random variable $X$ and constant $a > 0$

$$P(|X - E[X]| \geq a) \leq \frac{\text{var}[X]}{a^2}$$

• To rephrase, let $a = k\sigma$ where $\sigma = \text{std}[X]$

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

• In words, the probability that $X$ is greater than $k$ standard deviations from the mean is small
Proof of Chebyshev’s inequality