Recap

• (Ch 3) Random outcomes and events

Today

• (Ch 4) Random variables
Conditional probability and independence

• The definition of conditional probability is

\[ P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} \]

• If \( E_1 \) and \( E_2 \) are independent, then \( P(E_1 \cap E_2) = P(E_1)P(E_2) \), so

\[ P(E_2|E_1) = P(E_2) \]
Random variables

- A random variable is function that maps events to real numbers

- Example: Toss a coin. Let random variable $X$ be
  - 0 if the coin comes up heads
  - 1 if the coin comes up tails

Let $X = \begin{cases} 0 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \end{cases}$

- Random variables have nothing to do with variables in a program, even though we might use a programming variable to represent a random variable.
Random variables: more examples

• Number of pairs in a hand of 5 cards
  • Let a single outcome be the hand of cards
  • Each outcome maps to a number from 0 to 2

• Number of electoral votes that a US presidential candidate will win
  • Let a single outcome be the list of votes or non-votes of all registered voters
  • Each outcome maps to a number from 0 to 538

\[ \binom{52}{5} \]
Random variables and events

• Let $X$ be a random variable

• The set of outcomes \( \{ A : X(A) = x \} \) is an event with probability \( P(X = x) \)

• Likewise, the set of outcomes \( \{ A : X(A) \leq x \} \) is an event with probability \( P(X \leq x) \)
Random variables and events: dice example

• Roll 2 three-sided dice

• How many outcomes?

• Define the following random variables
  • Let $X$ be the value of die 1
  • Let $Y$ be the value of die 2
  • Let sum $S = X + Y$
  • Let difference $D = X - Y$
Random variables and events: dice example

- Calculate the following probabilities
  
  - $P(X = 1) = \frac{1}{3}$
  - $P(Y \leq 2) = \frac{2}{3}$
  - $P(S = 5) = \frac{2}{9}$
  - $P(D \leq -1) = \frac{1}{9} = \frac{1}{3}$
Probability distribution

• $P(X = x)$ is called the **probability distribution** of $X$

• $P(X = x)$ is also denoted as $P(x)$ or $p(x)$

• $P(X = x) \geq 0$ for all values that $X$ can take and is $0$ everywhere else

• The sum of the probability distribution $\sum_x P(x) = 1$ because
  • $\{A : X(A) = x_i\}$ and $\{A : X(A) = x_j\}$ are disjoint for $x_i \neq x_j$
  • $\{A : X(A) = x_i\}$ cover the sample space $\Omega$
Cumulative distribution function

- \( P(X \leq x) \) is called the **cumulative distribution function** of \( X \)

- \( P(X \leq x) \) is also denoted as \( f(x) \)

- \( P(X \leq x) \) is a non-decreasing function of \( x \)
Distribution functions: dice example

\[ P(S = s) = p(s) \]

\[ P(S \leq s) = f(s) \]
Joint probability distribution

• The **joint probability distribution** of two random variables $X$ and $Y$ is
  \[ P(\{X = x\} \cap \{Y = y\}) \], also denoted $P(x, y)$ for short

• We can recover the individual probability distributions $P(x)$ and $P(y)$ from the joint probability distribution as follows
  \[ P(x) = \sum_y P(x, y) \text{ and } P(y) = \sum_x P(x, y) \]

• The sum of the joint probability distribution $\sum_y \sum_x P(x, y) = 1$
Joint probability distribution: dice example

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
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<tr>
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<td>$\frac{1}{9}$</td>
<td>$\frac{1}{9}$</td>
</tr>
<tr>
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<td>$\frac{1}{9}$</td>
<td>$\frac{1}{9}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{9}$</td>
</tr>
</tbody>
</table>

$P(x, y)$

$P(x)$

$P(y)$
Joint probability distribution: dice example

\[ P(s, d) \]

\[
\begin{array}{cccccc}
& -2 & -1 & 0 & 1 & 2 \\
2 & 0 & 0 & \frac{1}{9} & 0 & 0 \\
1 & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 \\
0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{9} \\
-1 & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 \\
-2 & 0 & 0 & \frac{1}{9} & 0 & 0 \\
\end{array}
\]

\[ P(s) \]

\[
\begin{array}{cccc}
& \frac{1}{9} & \frac{3}{9} & \frac{1}{3} & \frac{2}{9} \\\n2 & \frac{1}{9} \\
1 & \frac{3}{9} \\
0 & \frac{1}{3} \\
-1 & \frac{2}{9} \\
-2 & \frac{1}{9} \\
\end{array}
\]
Independence of random variables

• Random variables $X$ and $Y$ are independent if

$$P(x, y) = P(x)P(y) \text{ for all } x \text{ and } y$$

• For the dice example, are the following variables independent?

  • $X$ and $Y$ Yes
  • $S$ and $D$ No e.g. $P(s=2, d=-2) \neq P(s=2)P(d=-2)$
Conditional probability distribution

- The **conditional probability distribution** of $X$ given $Y$ is

$$P(x|y) = \frac{P(x, y)}{P(y)}$$

- For any given $y$, \( \sum_x P(x|y) = 1 \)

- If $X$ and $Y$ are independent, $P(x, y) = P(x)P(y)$, so $P(x|y) = P(x)$
### Conditional distribution: dice example

The conditional distribution of rolling a dice can be expressed as:

$$P(s|d) = \frac{P(s, d)}{P(d)}$$

where:
- $s$ represents the outcome of the dice roll.
- $d$ represents the desired outcome.

The table below represents the joint distribution $P(s, d)$, where $s$ is the result of rolling the dice and $d$ is the desired outcome.

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
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<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
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<td>0</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The marginal distribution $P(s)$ is represented by the red circles.

The marginal distribution $P(d)$ is represented by the bottom row of the table.

The joint distribution $P(s, d)$ is represented by the grid.

The conditional distribution $P(s|d)$ is represented by the ratios in the table.
Bayes rule for random variables

- Bayes rule for events generalizes to random variables

\[ P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{P(y|x)P(x)}{\sum_x P(y|x)P(x)} \]

- Let’s check Bayes rule for a case of the dice example

\[
P(D = 0|S = 2) = \frac{P(S = 2|D = 0)P(D = 0)}{P(S = 2)} = 1\]