Recap

- (Ch 3) More on probability
  - How to count without counting
  - Independence

Today

- (Ch 3) How are events related?
  - Independence
  - Conditional Probability
  - Conditional Independence
Review of independence

• Two events are independent if knowing whether one event happened does not change the probability of the other event.

• Definition: Events $E_1$ and $E_2$ are independent if and only if

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$
Overbooking example 1

An airline has a flight with 6 seats. They always sell 7 tickets for this flight. If ticket holders show up independently with probability $p$ what is the probability that the flight is overbooked?

$$p\left(\text{all 7 show up}\right) = p \cdot p \cdots p = p^7$$
Overbooking example 2

An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability \( p \) what is the probability that exactly 6 passengers show up?

\[
P(\{\text{exactly 6 show up}\}) = \binom{8}{6} p^6 (1-p)^2
\]

- \( \binom{8}{6} \): number of ways of choosing the specific 6
- \( p^6 (1-p)^2 \): probability that a specific 6 show up and the other don't show up
Overbooking example 3

An airline has a flight with 6 seats. They always sell 8 tickets for this flight. If ticket holders show up independently with probability $p$ what is the probability that the flight is overbooked?

$$P(\{\text{overbooked}\}) = \binom{8}{2} p^2 (1-p) + \binom{8}{9} p^8$$

$$= 8 p^2 (1-p) + p^8$$
Overbooking example 4

An airline has a flight with $s$ seats. They always sell $t$ tickets for this flight. If ticket holders show up independently with probability $p$ what is the probability that exactly $u$ passengers show up?

$$P(\{\text{exactly } u \text{ show up}\}) = \binom{t}{u} p^u (1-p)^{t-u}$$
Overbooking example 5

An airline has a flight with $s$ seats. They always sell $t$ tickets for this flight. If ticket holders show up independently with probability $p$ what is the probability that the flight is overbooked?

$$P(\{\text{overbooked}\}) = \sum_{u=s+1}^{t} \binom{t}{u} p^u (1-p)^{t-u}$$
Pairwise independence is not independence

• Draw three cards from standard deck with replacement
  • Let $E_1$ be the event that card 1 and card 2 have the same suit
  • Let $E_2$ be the event that card 2 and card 3 have the same suit
  • Let $E_3$ be the event that card 3 and card 1 have the same suit

• Are the following events independent?
  • $E_1$ and $E_2$?
  • $E_2$ and $E_3$?
  • $E_3$ and $E_1$?
  • $E_1$, $E_2$ and $E_3$?

$P(E_1) = \frac{1}{4}$

\[
\begin{array}{c}
\checkmark \\
\checkmark \\
\checkmark \\
\times \\
\end{array}
\]

$E_1, E_2 \& E_3$ are pairwise independent

No, because if $E_1 \& E_2$ happened, then $E_3$ definitely happened.
Conditional probability

The **conditional probability** of $E_2$ given $E_1$ is the probability of $E_2$ given that $E_1$ has happened.

$$P(E_2 | E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$
Conditional probability: die example

Suppose you roll a six-sided die

- Let $E_1$ be the event that it comes up **either 1, 2 or 3**
- Let $E_2$ be the event that it comes up **even**

\[
P(E_2 | E_1) = \frac{1}{3}
\]
Bayes rule for events (simple form)

• The definition of conditional probability implies that

\[ P(E_2|E_1)P(E_1) = P(E_1 \cap E_2) = P(E_1|E_2)P(E_2) \]

• Bayes rule

\[ P(E_2|E_1) = \frac{P(E_1|E_2)P(E_2)}{P(E_1)} \]
Bayes rule: car example

There are two car factories, A and B, that supply the same dealer

- Factory A produced 1000 cars, of which 10 were lemons
- Factory B produced 2 cars and both were lemons

You bought a car that turned out to be a lemon. What is the probability that it came from factory B?

\[
P(B | L) = \frac{P(L | B) P(B)}{P(L)} = \frac{1 \cdot \frac{2}{1002}}{\frac{12}{1002}} = \frac{2}{12} = \frac{1}{6}\]
Total probability

\[ p(E_i) = p(E_i \cap E_2) + p(E_i \cap E_2^c) \]

\[ = p(E_i | E_2) p(E_2) + p(E_i | E_2^c) p(E_2^c) \]
Bayes rule for events (alternative form)

\[ P(E_2|E_1) = \frac{P(E_1|E_2)P(E_2)}{P(E_1)} = \frac{P(E_1|E_2)P(E_2)}{P(E_1|E_2)P(E_2) + P(E_1|E_2^C)P(E_2^C)} \]

**total probability**
Bayes rule: false positive example

Suppose there is a blood test for a rare disease.

- The disease occurs in 1 in every 10000 people
- If you have the disease, the test will say so with probability 0.95
- If you do not have it, the test will give a false positive with probability 0.001

What is \( p(D|T) \), the probability that you have the disease given that you have tested positive?

\[
p(D|T) = \frac{P(T|D) \cdot P(D)}{P(T|D) \cdot P(D) + P(T|D^c) \cdot P(D^c)} \approx 0.0094 < 1\% \]
Total probability (extended form)

If a set of disjoint events $E_i$ “cover” an event $A$, then the total probability formula is:

$$P(A) = \sum_i P(A|E_i)P(E_i)$$
Bayes rule for events (extended form)

If a set of disjoint events $E_i$ “cover” an event $A$, then Bayes rule is:

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{\sum_j P(A|E_j)P(E_j)}$$
Bayes rule: Monty Hall problem

Demo:
http://onlinestatbook.com/2/probability/monty_hall_demo.html
Bayes rule: Monty Hall problem

• Suppose you first select Door 1 and then Monty opens Door 2 to reveal a goat. Should you stay with Door 1 or switch to Door 3? (Note that this situation is equivalent to any other choices of doors)

• Define the following events
  • Let $C_i$ be the event that the car is behind Door $i$
  • Let $M_i$ be the event that Monty open Door $i$ to reveal a goat

• We will compare
  • Probability $P(C_1|M_2)$ of winning if you stay with Door 1
  • Probability $P(C_3|M_2)$ of winning if you switch to Door 3
Bayes rule: Monty Hall problem

• Stay with Door 1

\[ P(C_1|M_2) = \frac{P(M_2|C_1)P(C_1)}{P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2) + P(M_2|C_3)P(C_3)} = \frac{1}{3} \]

• Switch to Door 3

\[ P(C_3|M_2) = \frac{P(M_2|C_3)P(C_3)}{P(M_2|C_1)P(C_1) + P(M_2|C_2)P(C_2) + P(M_2|C_3)P(C_3)} = \frac{2}{3} \]
Conditional independence

Events \( E_1 \) and \( E_2 \) are **conditionally independent** given event \( A \) if

\[
P(E_1 \cap E_2 | A) = P(E_1 | A)P(E_2 | A)
\]