Recap

- (Ch 3) Basic ideas in probability
  - Outcomes, sample space and events
  - Probability axioms and properties
  - Using counting to determine probability

Today

- (Ch 3) More on probability
  - How to count without counting
  - Independence
  - Examples, examples, examples!

Why counting?

If all outcomes A_i in the sample space Ω have equal probability,

$$P(E) = \frac{number\ of\ outcomes\ in\ E}{total\ number\ of\ outcomes\ in\ \Omega} = \frac{|E|}{|\Omega|}$$

$|E|$ is the cardinality of E

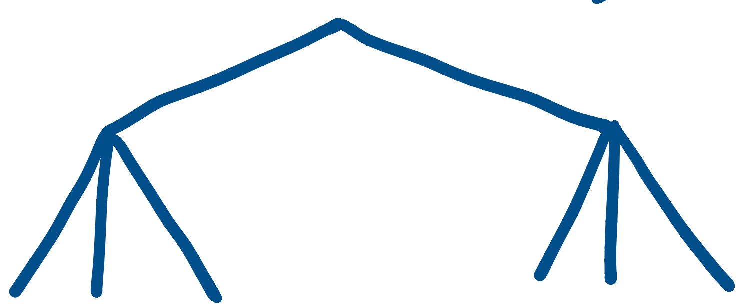
Multiplication principle

Suppose that a choice is made in two consecutive steps, such that:

- Step 1 has m choices
- Step 2 has n choices for each choice in step 2

Then the total number of combined choices is mn.

If $m=2$ and $n=3$, there are 6 choices altogether



step 1

step 2

Multiplication principle: examples

- How many Shakespearean insults are possible on http://insult.dream40.org?

58×71

adjectives

nouns

- How many ways are there to draw two cards of the same suit from a standard deck of 52 cards? The draws are without replacement.

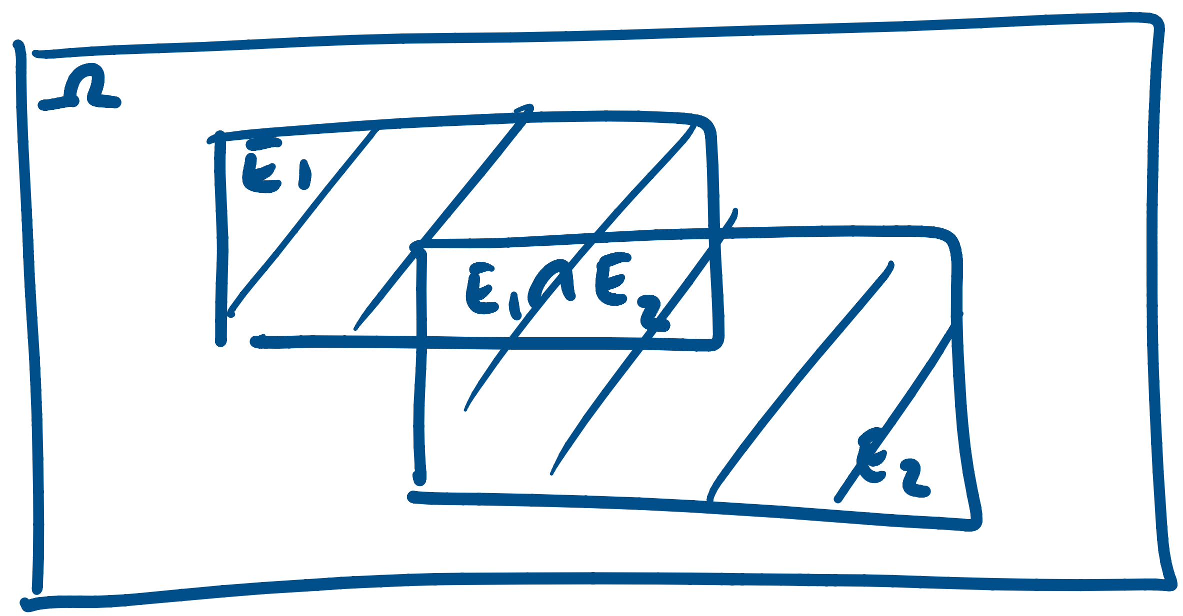
52×12

no. of same suit cards still in the deck

Inclusion-exclusion principle

Overcount and then subtract what you counted twice

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$



Inclusion-exclusion principle: example

How many ways are there to draw two cards of the same color or two cards of the same value ("pair") from a standard deck? The draws are without replacement.

$$|\{\text{same color}\}| = 52 \times 25$$

$$|\{\text{same val}\}| = 52 \times 3$$

$$|\{\text{same color}\} \cap \{\text{same val}\}| = 52 \times 1$$

$$|\{\text{same color}\} \cup \{\text{same val}\}| = 52(25 + 3 - 1)$$

$$= 52 \times 27$$

Permutations (order is important)

- In the 2016 Olympics, there were 9 sprinters in the mens 100m final. How many possible finishing orders were there?

$$9 \times 8 \times 7 \times \dots \times 1 = 9! = 362880$$

- How many ways to award gold, silver and bronze medals were there?

$$9 \times 8 \times 7 = \frac{9!}{6!}$$

$\leftarrow$ no. of permutations of all 9

$\leftarrow$ no. of permutations of the 6 non-medallists

$$= 504$$

Combinations (order is not important)

From among the 9 sprinters, how many ways are there to choose an all-star relay team of 4?

$$\frac{9 \times 8 \times 7 \times 6}{4!}$$

$9 \times 8 \times 7 \times 6$ ← no. of permutations of 4 out of 9

$4!$ ← no. of permutations of 4 members

$$= \frac{9!}{5!\,4!} = \binom{9}{4} = \binom{9}{5} = 126$$

Defn: $\binom{N}{k} = \frac{N!}{k!\,(N-k)!} = \binom{N}{N-k}$ "N choose k"

Generalized permutations/combinations

- At the FIFA world cup, 32 teams are grouped into 8 groups of 4 teams, but the home team is always in Group A. How many ways are there?

$$\frac{31!}{3!\,4!\,4!\,4!\,4!\,4!\,4!\,4!}$$

- How many ways are there to rearrange the letters of ILLINOIS?

$$\frac{8!}{\underbrace{3!}_{\text{Is}}\,\underbrace{2!}_{\text{Ls}}\,1!\,1!\,1!}$$

Probability examples

- What is the probability of drawing two cards of the same color or two cards of the same value ("pair") from a standard deck? The draws are without replacement.

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\cancel{52} \cdot 27}{\cancel{52} \cdot 51} = \frac{9}{17}$$

- Assuming all outcomes are equiprobable, what is the probability of Usain Bolt being on the all-star relay team?

$$P(E) = \frac{|E|}{|\Omega|} = \frac{\binom{8}{3}}{\binom{9}{4}} = \frac{56}{126} = \frac{4}{9}$$

Birthday problem

Among 30 people, what is the probability that at least 2 of them celebrate their birthday on the same day? Assume that there is no February 29 and each day of the year is equally likely to be a birthday.

$$P(\{\text{at least 2 same}\}) = 1 - P(\{\text{all different}\})$$

$$= 1 - \frac{365 \cdot 364 \cdot \cdots 336}{365^{30}} = 1 - \frac{365!}{335!} \frac{1}{365^{30}}$$

$$= 0.706$$

70%

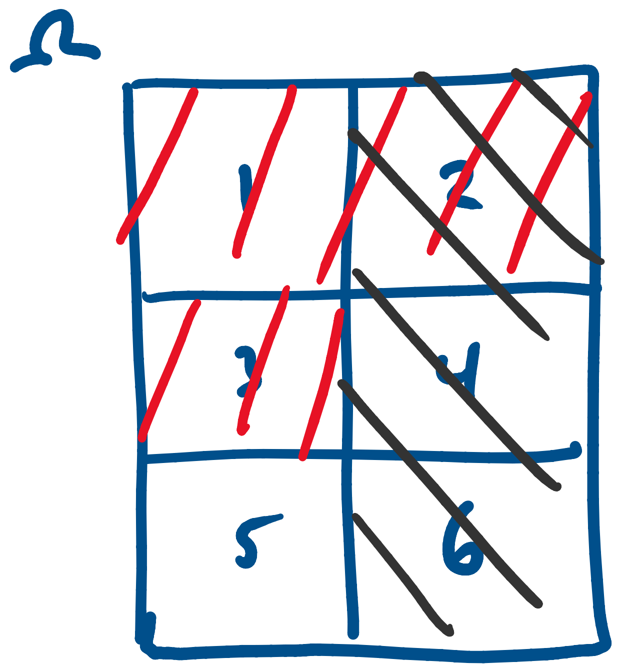
Independence

- Two events are independent if knowing whether one event happened does not change the probability of the other event.

- Examples
  - Tossing a nickel is independent of tossing a dime
  - When drawing two socks from a bag with replacement, the drawings are independent
  - When drawing two socks from a bag without replacement, the drawings are dependent (unless the socks are all the same color)

Dependence: die example

Suppose you roll a six-sided die. The event that it comes up $\underbrace{\textbf{either 1, 2 or 3}}_{P(E_1) = \frac{1}{2}}$ is dependent on the event that it comes up $\underbrace{\textbf{even}}_{P(E_2) = \frac{1}{2}}$.



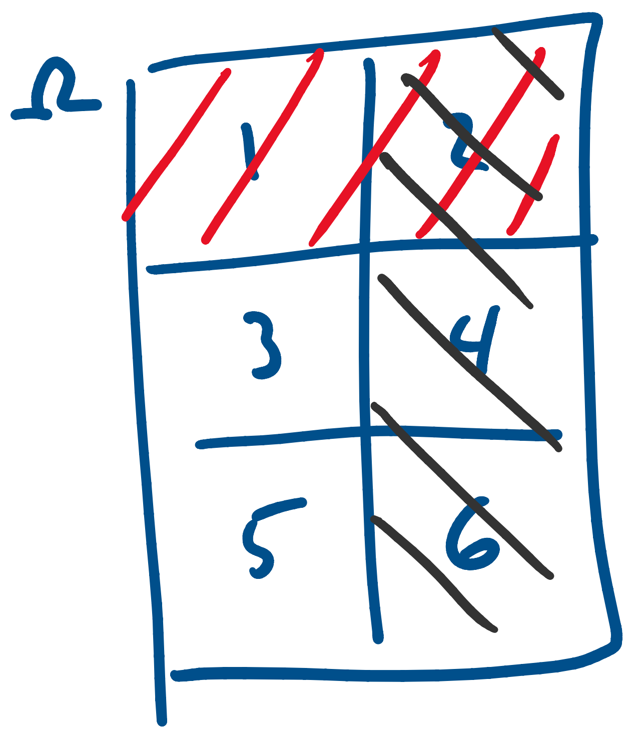
$$P(E_1 \cap E_2) = \frac{1}{6}$$
$$\neq P(E_1)\,P(E_2)$$

Independence: die example

Suppose you roll a six-sided die. The event that it comes up **either 1 or 2** is independent of the event that it comes up **even**.

$P(E_1) = \frac{1}{3}$

$P(E_2) = \frac{1}{2}$



$$P(E_1 \cap E_2) = \frac{1}{6}$$

$$= P(E_1)P(E_2)$$

Definition of independence

- Events E_1 and E_2 are **independent** if and only if

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

- Probability of rolling a six-sided die and getting 3 sixes in a row

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$

Testing for independence

Suppose you draw one card from a standard deck.

- E_1 is the event that the card is a King, Queen or Jack
- E_2 is the event that the card is a Heart

Are E_1 and E_2 independent?

$$P(E_1) = \frac{12}{52} = \frac{3}{13} \qquad P(E_2) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_1 \cap E_2) = \frac{3}{52} = P(E_1)P(E_2)$$

So, E_1 & E_2 are independent