Recap

• (Ch 3) Basic ideas in probability
  • Outcomes, sample space and events
  • Probability axioms and properties
  • Using counting to determine probability

Today

• (Ch 3) More on probability
  • How to count without counting
  • Independence
  • Examples, examples, examples!
Why counting?

If all outcomes \( A_i \) in the sample space \( \Omega \) have equal probability,

\[
P(E) = \frac{\text{number of outcomes in } E}{\text{total number of outcomes in } \Omega} = \frac{|E|}{|\Omega|}
\]

\(|E|\) is the cardinality of \( E \)
Multiplication principle

Suppose that a choice is made in two consecutive steps, such that:
- Step 1 has $m$ choices
- Step 2 has $n$ choices for each choice in step 2

Then the total number of combined choices is $mn$.

If $m=2$ and $n=3$, there are 6 choices altogether.

step 1

step 2
Multiplication principle: examples

• How many Shakespearean insults are possible on http://insult.dream40.org?

• How many ways are there to draw two cards of the same suit from a standard deck of 52 cards? The draws are without replacement.
Inclusion-exclusion principle

Overcount and then subtract what you counted twice

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$
Inclusion-exclusion principle: example

How many ways are there to draw two cards of the same color or two cards of the same value ("pair") from a standard deck? The draws are without replacement.

\[
\begin{align*}
|\{\text{same color}\}| &= 52 \times 25 \\
|\{\text{same val}\}| &= 52 \times 3 \\
|\{\text{same color}\} \cap \{\text{same val}\}| &= 52 \times 1 \\
|\{\text{same color}\} \cup \{\text{same val}\}| &= 52(25 + 3 - 1) = 52 \times 27
\end{align*}
\]
Permutations (order is important)

• In the 2016 Olympics, there were 9 sprinters in the mens 100m final. How many possible finishing orders were there?

\[ 9 \times 8 \times 7 \times \ldots \times 1 = 9! = 362,880 \]

• How many ways to award gold, silver and bronze medals were there?

\[ 9 \times 8 \times 7 = \frac{9!}{6!} \leftarrow \text{no. of permutations of all 9} \]

\[ = \frac{362,880}{720} \leftarrow \text{no. of permutations of the 6 non-medallists} \]

\[ = 504 \]
Combinations (order is not important)

From among the 9 sprinters, how many ways are there to choose an all-star relay team of 4?

\[
\frac{9 \times 8 \times 7 \times 6}{4!} \quad \text{no. of permutations of 4 out of 9}
\]

\[
= \frac{9!}{5!4!} = \binom{9}{4} = \binom{9}{5} = 126
\]

Define: \( \binom{N}{h} = \frac{N!}{h!(N-h)!} = \binom{N}{N-h} \) "N choose h"
Generalized permutations/combinations

• At the FIFA world cup, 32 teams are grouped into 8 groups of 4 teams, but the home team is always in Group A. How many ways are there?

\[
\frac{31!}{3! \cdot 4! \cdot 4! \cdot 4! \cdot 4! \cdot 4! \cdot 4!}
\]

• How many ways are there to rearrange the letters of ILLINOIS?

\[
\frac{8!}{3! \cdot 2! \cdot 1! \cdot 1! \cdot 1! \cdot 1!}
\]
Probability examples

• What is the probability of drawing two cards of the same color or two cards of the same value (“pair”) from a standard deck? The draws are without replacement.

\[ P(E) = \frac{\binom{13}{1}}{\binom{52}{2}} = \frac{\frac{52 \times 27}{2}}{\frac{52 \times 51}{2}} = \frac{9}{17} \]

• Assuming all outcomes are equiprobable, what is the probability of Usain Bolt being on the all-star relay team?

\[ P(E) = \frac{\binom{13}{1}}{\binom{52}{1}} = \frac{\frac{9 \times 10}{2}}{\frac{9 \times 10}{2}} = \frac{5 \times 6}{12} = \frac{4}{9} \]
Birthday problem

Among 30 people, what is the probability that at least 2 of them celebrate their birthday on the same day? Assume that there is no February 29 and each day of the year is equally likely to be a birthday.

\[
P(\{ \text{at least 2 same} \}) = 1 - P(\{ \text{all different} \})
\]

\[
= 1 - \frac{365 \cdot 364 \cdot \ldots \cdot 336}{365^{30}}
\]

\[
= 0.706 \quad (70\%)
\]
Independence

• Two events are independent if knowing whether one event happened does not change the probability of the other event.

• Examples
  • Tossing a nickel is independent of tossing a dime
  • When drawing two socks from a bag with replacement, the drawings are independent
  • When drawing two socks from a bag without replacement, the drawings are dependent (unless the socks are all the same color)
Dependence: die example

Suppose you roll a six-sided die. The event that it comes up **either 1, 2 or 3** is dependent on the event that it comes up **even**.

\[ P(E_1) = \frac{1}{2} \]

\[ P(E_2) = \frac{1}{2} \]

\[ P(E_1 \cap E_2) = \frac{1}{6} \]

\[ \neq P(E_1) \cdot P(E_2) \]
Independence: die example

Suppose you roll a six-sided die. The event that it comes up either 1 or 2 is independent of the event that it comes up even.

\[ p(E_1) = \frac{1}{2} \]

\[ p(E_2) = \frac{1}{2} \]

\[ p(E_1 \cap E_2) = \frac{1}{6} = p(E_1) p(E_2) \]
Definition of independence

• Events $E_1$ and $E_2$ are independent if and only if

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

• Probability of rolling a six-sided die and getting 3 sixes in a row

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}$$
Testing for independence

Suppose you draw one card from a standard deck.

- $E_1$ is the event that the card is a King, Queen or Jack
- $E_2$ is the event that the card is a Heart

Are $E_1$ and $E_2$ independent?

\[
\begin{align*}
P(E_1) &= \frac{12}{52} = \frac{3}{13} \\
P(E_2) &= \frac{13}{52} = \frac{1}{4} \\
P(E_1 \cap E_2) &= \frac{3}{52} = P(E_1)P(E_2)
\end{align*}
\]

So, $E_1$ & $E_2$ are independent