# CS 361 Sample Midterm 1

**NAME:** SOLUTIONS

**NETID:**

**CIRCLE YOUR DISCUSSION:**

**Thu 2-3 Thu 4-5 Fri 10-11 Fri 11-12**

- Be sure that your exam booklet has 6 pages including this cover page
- Make sure to write your name exactly as it appears on your i-card
- Write your netid and circle your discussion section on this page
- **Show your work**
- This is a closed book exam
- You are allowed one handwritten 8.5 x 11-inch sheet of notes (both sides)
- You may **not** use a calculator or any other electronic device
- Turn off your phone and store it in your backpack
- Store away any other electronic devices including earphones and smartwatches
- Absolutely no interaction between students is allowed
- Use backs of pages for scratch work if needed
- Show your i-card when handing in your exam

| Problem | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|
| Possible | 30 | 30 | 30 | 30 | 30 | 150 |
| Score | | | | | | |

**Problem 1 (30 pts)**

1. (10 points) Say you score 40 points out of 50 on Homework 1. Then you score 50 out of 50 on each of the remaining 9 homework assignments. Are your homework scores symmetric, left-skewed or right-skewed? Justify your answer with calculation.

   Circle one answer: SYMMETRIC LEFT-SKEWED RIGHT-SKEWED

   $$\text{mean} = \frac{49 + 9(50)}{10} = 49$$

   median = 50 > mean, so left-skewed

2. (20 points) Let $\{x\}$ be a dataset consisting of $N$ real numbers, $x_1, \ldots, x_N$. Show that the function $f(\mu) = \sum_i (x_i - \mu)^2$ is minimized when $\mu = \text{mean}(\{x\})$.

   We will set $f'(\mu) = 0$

   $$0 = \sum_i \left(-2(x_i - \mu)\right) = 2\left(N\mu - \sum_i x_i\right)$$

   So $N\mu = \sum_i x_i$

   $$\mu = \frac{1}{N} \sum_i x_i = \text{mean}(\{x\})$$

   Also, $f''(\mu) = 2N > 0$, so $f(\mu)$ is convex

   $\therefore$ $\mu = \text{mean}(\{x\})$ minimizes $f(\mu)$

**Problem 2 (30 pts)** Suppose a teacher gives a multiple choice test of 10 questions to $N$ students. Let $x_i$ and $y_i$ be the number of questions that the $i$th student gets right and wrong, respectively.

1. (10 points) Write a formula for mean($\{y\}$) in terms of mean($\{x\}$). Draw a box around your answer.

Since $y_i = 10 - x_i$,

$$\boxed{\text{mean}(\{y\}) = 10 - \text{mean}(\{x\})}$$

2. (10 points) Write a formula for std($\{y\}$) in terms of std($\{x\}$). Draw a box around your answer.

$$\boxed{\text{std}(\{y\}) = \text{std}(\{x\})}$$

3. (10 points) Show that corr($\{(x, y)\}$) $= -1$

$$\hat{y}_i = \frac{y_i - \text{mean}(\{y\})}{\text{std}(\{y\})} = \frac{(10 - x_i) - (10 - \text{mean}(\{x\}))}{\text{std}(\{x\})}$$

$$= -\frac{x_i - \text{mean}(\{x\})}{\text{std}(\{x\})} = -\hat{x}_i$$

So $\text{corr}(\{(x,y)\}) = \frac{1}{N}\sum_i \hat{x}_i \hat{y}_i = -\frac{1}{N}\sum_i \hat{x}_i^2$

$$= -\frac{1}{N}\sum_i (\hat{x}_i - 0)^2 = -\text{var}(\hat{x}_i) = -1$$

since $\text{mean}(\{\hat{x}_i\}) = 0$ since $\text{var}(\{\hat{x}_i\}) = 1$

**Problem 3 (30 pts)**

1. (15 points) You draw a card uniformly at random from a standard deck. Let $R$ be the event that the card is red and let $B$ be the event the card is black. Are $R$ and $B$ independent? Draw a box around your answer. Justify your answer with calculation.

$$P(R) = \frac{26}{52} = \frac{1}{2} \qquad P(B) = \frac{26}{52} = \frac{1}{2}$$

$$P(R \cap B) = 0 \neq P(R)P(B)$$

$\boxed{\text{So, } R \text{ \& } B \text{ are not independent}}$

2. (15 points) A student takes a multiple choice test. Each question has 4 answer choices. If the student knows the answer to a question, the student gives the right answer. Otherwise, the student guesses uniformly at random from among the answer choices. The student knows the answer to 60% of the questions. Let $K$ be the event that student knows the answer to a question and let $C$ be the event that the students answers that question correctly. Calculate $P(K|C)$. Draw a box around your answer.

$$P(K|C) = \frac{P(C|K)P(K)}{P(C|K)P(K) + P(C|K^c)P(K^c)}$$

$$= \frac{(1)(0.6)}{(1)(0.6) + (0.25)(0.4)}$$

$$= \frac{0.6}{0.6 + 0.1} = \boxed{\frac{6}{7}}$$

**Problem 4 (30 pts)**

1. (15 points) Suppose you make a bet on the roll of a fair six-sided die. If the die comes up 6, you win \$60. Otherwise, you lose \$30. Let random variable $W$ be your net winnings. Calculate $E[W]$. Draw a box around your answer.

$$E[W] = \frac{1}{6}60 + \frac{5}{6}(-30) = 10 - 25 = \boxed{-15}$$

2. (15 points) Suppose you make the same bet as above three times in a row. Let random variable $T$ be your total net winnings after these three bets. Calculate $P(T \geq 0)$. You do not need to simplify your answer. Draw a box around your answer.

For the event $\{T < 0\}$ you must lose all 3 times.

So $P(T<0) = \left(\frac{5}{6}\right)^3$

$$\boxed{\text{So } P(T \geq 0) = 1 - \left(\frac{5}{6}\right)^3}$$

**Problem 5 (30 pts)**

1. (15 points) If I attempt 20 free throws, what is the probability that I score more than 10 points? Assume that each free throw attempt is independent. Also assume that on each free throw attempt, I earn 1 point with probability $p$ and 0 points with probability $1 - p$. You may use choose and summation notation in your answer. Draw a box around your answer.

$$\boxed{\sum_{u=11}^{20} \binom{20}{u} p^u (1-p)^{20-u}}$$

2. (15 points) Suppose the daily number of car accidents per mile of road is a Poisson random variable with intensity $\lambda = 0.001$. What is the probability that there are 10 accidents on 1000 miles of road in a week? You do not need to simplify your answer. Draw a box around your answer.

For 1000 miles and 7 days,

$$\lambda = 0.001 \times 1000 \times 7 = 7$$

So probability of 10 accidents

$$= \boxed{\frac{e^{-7}\, 7^{10}}{10!}}$$