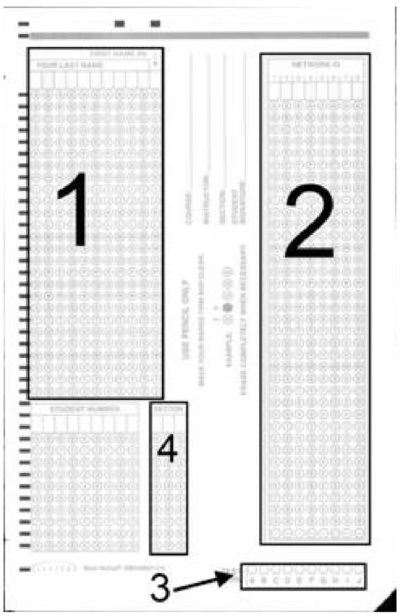
CS357 **Summer 2013**

Final Exam
Friday, August 2^{nd}

READ and complete the following:

- Bubble your Scantron only with a No. 2 pencil.
- On your Scantron (shown in the figure below), bubble :

  1. Your Name
  2. Your NetID
  3. Form letter "**A**"



- No electronic devices or books are allowed while taking this exam. However, you may use two "cheat sheets" - each sheet of size 8.5" x 11" or smaller.
- Please fill in the most correct answer on the provided Scantron sheet.
- We will not answer any questions during the exam.
- Each question has only ONE correct answer.
- You must stop writing when time is called by the proctors.
  No extra time will be given after the exam ends to fill in bubble sheets with answers.
- Hand in both these exam pages and the Scantron.
- DO NOT turn this page UNTIL the proctor instructs you to.

1. What is the Taylor Series expansion of the function $f(x, y) = e^{-(x^2+y^2)}$ about $(x, y) = (0, 0)$ for terms $x^{k_1}y^{k_2}$ where $|k| = 0, 1, 2, 3, 4$ and $|k| = k_1 + k_2$?

(a) $x^2 + y^2 + x^4 + y^4$

(b) $-x^2 - y^2 + x^4 + y^4$

(c) $1 - x^2 - y^2 + \frac{1}{2}x^4 + \frac{1}{2}y^4$

(d) $1 - x^2 - y^2 + \frac{1}{2}x^4 + x^2y^2 + \frac{1}{2}y^4$

2. Suppose A is an $m \times n$ matrix and A has the QR factorization $A = QR$. Which one of the following three properties is **NOT TRUE**? If all are **TRUE**, select (d).

(a) $Q^TQ = I$, where I is the identity matrix

(b) $\|Qv\|_2 = \|v\|_2$

(c) $Q^T = Q^{-1}$

(d) All are True.

3. What is the asymptotic best bound for the cost of performing Gaussian Elimination for an $n \times n$, 5-band matrix (i.e. $m = 5$)?

(a) $\mathcal{O}(25n)$

(b) $\mathcal{O}(5n^2)$

(c) $\mathcal{O}(25n^2)$

(d) $\mathcal{O}(n^3)$

4. If A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 8 \\ 3 & 8 & 11 \end{bmatrix}$ is expressed in the form LL^T by Cholesky factorization, what is L?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}$

5. Which of the following statements is **NOT TRUE** concerning the condition number $\kappa(A)$ of a square matrix A?

(a) $\kappa(A)$ measures the ratio of the largest stretching to the smallest stretching of any non-zero vector by the matrix A.

(b) $0 \leq \kappa(A) \leq 1$

(c) $\kappa(A)$ indicates the sensitivity of the solution of $Ax = b$ to perturbations in A and b.

(d) If $\kappa(A) = \infty$ then A is singular.

6. **(True/False)** If $\hat{x}$ is the numerical solution to $Ax = b$, then the norm of the residual is defined as $\|b - A\hat{x}\|$. A small residual implies that $\|\hat{x} - x\|$ is also small.

(a) True

(b) False

7. In solving the system of equations $Ax = b$, for a matrix with $\kappa(A) \sim 10^6$ the elements of the solution vector will have approximately how many correct digits?
Hint: In MATLAB, the machine epsilon, $\epsilon_m \approx 2.2X10^{-16}$.

(a) 16

(b) 6

(c) 10

(d) Solution will be very inaccurate.

8. For $A = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 8 & 6 \\ 3 & 6 & 5 \end{bmatrix}$ you can write $LU = PA$ with partial pivoting. Which of the following would be the U and P?

(a) U= $\begin{bmatrix} 6 & 8 & 6 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$; P= $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) U= $\begin{bmatrix} 6 & 8 & 6 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$; P= $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) U= $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$; P= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) U= $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$; P= $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Which of the following is **NOT TRUE** for Cholesky factorization ($A = L * L^T$) of a matrix?

(a) Cholesky factorization does not require pivoting.

(b) It requires half as many flops as LU factorization.

(c) Cholesky factorization is possible for all symmetric matrices.

(d) The matrix L is lower triangular.

10. Consider a system of equations $Ax = b$, where A is an $n \times n$ matrix, in which the Conjugate Gradient (CG) iterative method is to be run and for which the function $\phi(x)$ is defined below.

$$\phi(x) = \frac{1}{2}x^T Ax - x^T b$$

Given this information which of the following is **NOT TRUE**?

(a) In our derivation of CG, we required that A has to be a symmetric and positive definite matrix.

(b) CG is based on the minimization of $\phi(x)$.

(c) For each new iteration x_{k+1} we choose $x_{k+1} = x_k + \alpha * r_k$ for some optimal choice of $\alpha \in \mathbb{R}$.

(d) The residual of the system, $r_k = (b - Ax_k)$ is equal to $- \bigtriangledown \phi(x_k)$.

11. What is the 2-norm of the matrix A with values shown below?

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

(a) 2

(b) 3

(c) 4

(d) 6

12. Suppose you are seeking the solution x^* to the system of equations $Ax = b$ and you arrive at an approximation $\hat{x}$ to x^*. With the error e and residual r terms defined below,

$$e = x^* - \hat{x}$$

$$r = b - A\hat{x}$$

which of the following relations is correct?

(a) $r = e$

(b) $Ar = e$

(c) $Ae = r$

(d) $Ar = b$

13. The Gauss-Seidel iterative method for solving $Ax = b$ with

$$A = \begin{bmatrix} 7 & 3 & 2 & 0 \\ 0 & -8 & 5 & 1 \\ 2 & 0 & 6 & 3 \\ 4 & 9 & 2 & 16 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 9 \\ 29 \\ 88 \end{bmatrix}$$

would converge for *any* initial guess.

(a) True

(b) False

14. Suppose A is given as

$$A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 3 & 3 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 2 & 9 & 0 & 0 \end{bmatrix}$$

Which sparse matrix storage structure we studied in lecture matches the following format?

$$AA = \begin{bmatrix} 1 & 5 & 3 & 3 & 2 & 6 & 2 & 9 \end{bmatrix}$$
$$JR = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 4 & 4 \end{bmatrix}$$
$$JC = \begin{bmatrix} 1 & 4 & 1 & 2 & 3 & 4 & 1 & 2 \end{bmatrix}$$

(a) MSR

(b) CSR

(c) COO

(d) Dense

15. **(True/False)** The Trapezoid Rule can **always** integrate a first degree polynomial exactly.

(a) True

(b) False

16. For composite integration rules which of the following gives the smallest asymptotic error bound (using $O(h^p)$ where $h = \frac{b-a}{n}$)?

(a) Boole's
(b) Simpson's 3/8
(c) Trapezoid
(d) Simpson's 1/3

17. Using Newton's method, find the value of $\frac{1}{\sqrt[3]{7}}$ by using an initial guess $x_0 = 1$. What is the result of a single iteration x_1?

(a) 6/7

(b) 5/7

(c) 4/7

(d) 1/2

18. A 4-point Gauss Quadrature scheme will exactly integrate (up to rounding error) **any** polynomial of degree less than or equal to _____.

(a) 4
(b) 5
(c) 6
(d) 7

19. Apply **one** iteration of the normalized Inverse Power Method to the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$
to find an eigenvector belonging to the **smallest** eigenvalue of A. As a starting vector use using $x_0 = \begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$
After normalization using the ∞-norm, one entry of the resulting vector will be negative one. What is the value of the other entry?
Hint: you will need to compute A^{-1}.

(a) 0
(b) 12/21
(c) 1/2
(d) 1/3

20. Consider A $= \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$, b $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Using Gauss-Seidel iterative method to approximate the solution $Ax = b$ with a starting guess of $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. what is x_1?

- **(a)** $x_1 = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix}$
- **(b)** $x_1 = \begin{bmatrix} 1/3 \\ -1/2 \end{bmatrix}$
- **(c)** $x_1 = \begin{bmatrix} 1/3 \\ -1/15 \end{bmatrix}$
- **(d)** $x_1 = \begin{bmatrix} 1/2 \\ 1/5 \end{bmatrix}$

21. Suppose $x_1, x_2, ..., x_5$ are distinct and that $y_1, y_2, ..., y_5$ are given. There is a unique polynomial $p(x)$ of degree at most _____ such that $p(x_i) = y_i$.

- **(a)** 4
- **(b)** 5
- **(c)** 6
- **(d)** 10

22. A Newton form of the quadratic interpolant of $f(x) = \frac{20}{x+1}$ using $x = 0, 1, 4$ is

- **(a)** $p_2(x) = 20x - 10(x-1) + 2(x-4)$
- **(b)** $p_2(x) = 20 - 10x + 2x(x-1)$
- **(c)** $p_2(x) = 20 + 10x + 2x(x-1)$
- **(d)** $p_2(x) = 5(x-1)(x-4) - \frac{10}{3}x(x-4) + \frac{1}{4}x(x-1)$

23. What is the numerical approximation to

$$\int_{-1}^{1} x^8 dx$$

using 2-pt Gaussian Quadrature? (the Guassian nodes are $\pm\frac{1}{\sqrt{3}}$ and the corresponding weights are both 1.

- **(a)** 2/27
- **(b)** 2/81
- **(c)** 2/9
- **(d)** 2/3

24. What is the column rank of the matrix A, when we know its SVD factorization shown below?

$$A = U * \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} * V^T$$

(a) 3

(b) 4

(c) 6

(d) Insufficient information is given in order to determine the rank of A.

25. Find the solution x to the Linear Least Squares problem $Ax \approx b$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 1/2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

(a) $x = \begin{bmatrix} 2 \\ 1/2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

(b) $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(d) $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Extra Credit

Answering, the question below, correctly will add points to your exam total. Answering incorrectly or not answering will not add points to your exam total.

26. Given the code below, compute the BEST asymptotic bound for the number of floating point operations needed to compute the following nested loops based on the value `n`.

```
for k = 1:(n-1)
    for i = (k+1):n
        xmult = a(i,k)/a(k,k);
        a[i,k] = xmult;
        for j = (k+1):n
            a(i,j) = a(i,j) - (xmult)*a(k,j);
        end
        b(i) = b(i) - (xmult)* b(k);
    end
end
```

(a) $\mathcal{O}(n)$

(b) $\mathcal{O}(n^2)$

(c) $\mathcal{O}(n^3)$

(d) $\mathcal{O}(n^4)$