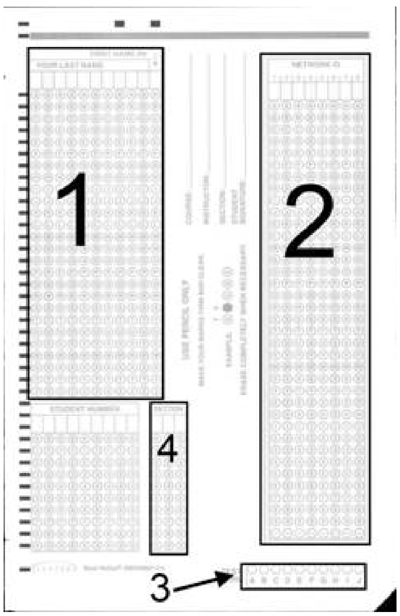
CS357 **Spring 2012**
Final Exam (180 minutes) **Friday, May 4th**
Form A

READ and complete the following:

- Bubble your Scantron only with a No. 2 pencil.
- On your Scantron (shown in the figure below), bubble :

  1. Your Name
  2. Your NetID
  3. Form letter "**A**"



- This exam has 50 questions and 2 extra credit questions.
- No electronic devices or books are allowed while taking this exam. However, you may use up to three "cheat sheets" - pages of size 8.5" x 11" or smaller.
- Please fill in the most correct answer on the provided Scantron sheet.
- We will not answer any questions during the exam.
- Each question has only ONE correct answer.
- You must stop writing when time is called by the proctors.
- Hand in both these exam pages and the Scantron.
- DO NOT turn this page UNTIL the proctor instructs you to.

1. What is the output of the following Matlab script?

```
a=1.0;
while ( a + 1.0 )>1.0
  a = a/2.0 ;
end
disp (2.0*a) ;
```

(a) Machine epsilon

(b) The smallest representable subnormal floating-point number.

(c) The smallest representable normalized floating-point number.

(d) 2×Machine epsilon

2. In IEEE double precision floating-point arithmetic, $3 + 2 \times e_m$ is equal to 3. (where e_m is the machine epsilon)

(a) True

(b) False

3. Let $x \in \mathbb{R}$ and $fl(x)$ denote the IEEE double precision floating point representation of x. Then $|x - fl(x)|$ gives the absolute value of the truncation error.

(a) True

(b) False

4. Consider using Newton's method to find the value of $1/\sqrt{5}$. By using an initial guess of $1/2$, what is the result of a single iteration x_1?

(a) $1/4$

(b) $9/20$

(c) $8/20$

(d) $5/16$

5. Given a function $\{(x, y) \in \mathbb{R}^2 \mid y = x^2 - \sqrt{x} - 4 \text{ and } x \geq 0\}$. Which of the following statements is correct?

(a) The function is injective.

(b) The function is surjective.

(c) The function is bijective.

(d) The function is neither injective nor surjective.

6. Using the definition of addition and multiplication of interval numbers, we have,

$$[-2, 3] * ([-3, 1] + [2, 3]) = [-2, 3] * [-3, 1] + [-2, 3] * [2, 3]$$

(a) True

(b) False

7. The minimum distance ($|x - y|$) between two IEEE double precision floating point numbers, x not equal to y, with both x and y belong to $[1, 10)$ is equal to the Machine epsilon.

(a) True

(b) False

8. What is the complexity of solving $Ax = b$ given an LU factorization of A? You may assume that A is an $n \times n$ matrix and b is an $n \times 1$ vector.

(a) $O(n^0)$

(b) $O(n^1)$

(c) $O(n^2)$

(d) $O(n^3)$

9. Which of the following properties is **NOT TRUE** of the bisection method for finding a root of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$?

(a) The number of iterations required to attain a given accuracy depends on the particular function.

(b) Its convergence rate is linear.

(c) It adds an equal number of digits of accuracy per iteration.

(d) It always converges to a root.

10. Suppose you applied an iterative numerical method for finding the roots of a function $f(x^*) = 0$ where $f : \mathbb{R} \to \mathbb{R}$, starting from an initial guess of x_0. The error in the approximate solution at the k^{th} step, $e_k = |x_k - x^*|$, is given by the following sequence,

k	1	2	3	4
e_k	10^{-2}	10^{-4}	10^{-8}	10^{-16}

What is the rate of convergence?

(a) Linear

(b) Super linear (but less than quadratic)

(c) Quadratic

(d) Cubic

11. You are given a function $f : \mathbb{R} \to \mathbb{R}$ where both the function f and it's derivative are continuous on $\mathbb{R}$. When Newton's method converges in finding a root $f(x) = 0$, it always converges at a quadratic rate of convergence.

(a) True

(b) False

12. In finding a root using the bisection method, we need to determine whether a root exists on a subdivided interval $[x_{left}, x_{right}]$. It's best numerically to check the sign of $f(x_{left}) * f(x_{right})$ to determine whether a root lies in the interval $[x_{left}, x_{right}]$.

(a) True

(b) False

13. Given that $f(x, y) = x^2 + y^2$, what is the Hessian matrix H at $x = 1, y = 1$?

(a) $H = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(c) $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

14. Why is it desirable to know the condition number of a problem?

(a) By choosing a different numerical method we can reduce the condition number and we will always produce accurate results.

(b) By choosing a different numerical method we can increase the condition number and we will always produce accurate results.

(c) A large condition number means that the problem is relatively insensitive to changes in the input data.

(d) A small condition number means that the problem is relatively insensitive to errors made during the computation.

For the next two questions, use the following matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{bmatrix}$$

15. For which values of k will the system $Ax = [2, 3, 7]^T$have a unique solution?

(a) $k \neq 2$

(b) $k \neq 3$

(c) $k \neq 5$

(d) $k \neq 7$

16. For $k = 5$ the above matrix A will **NOT** have an LU factorization.

(a) True

(b) False

17. What are the eigenvalues of the given matrix A?

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}$$

(a) $\lambda_1 = -1, \lambda_2 = 2$

(b) $\lambda_1 = 2, \lambda_2 = 5$

(c) $\lambda_1 = 3, \lambda_2 = 4$

(d) The eigenvalues are complex numbers.

18. You are given $n \times n$ matrices A, B and C along with an $n \times 1$ vector b. Assuming that a unique solution of,

$$A(B+C)^{-1}x = b$$

exists, to solve for x without computing an inverse of any matrix we would,_____(fill in the blank)____.

(a) use LU factorization to solve $(B+C)y = b$ then use LU factorization to solve $x = Ay$

(b) use LU factorization to solve $Ay = b$ then compute $x = (B+C)y$

(c) compute the solution directly as $x = (B+C)A^T b$

(d) need to compute the inverse of $(B+C)^{-1}$. The solution for x cannot be obtained without computing this inverse

19. We perform LU factorization of matrix A without pivoting, step by step by using elementary elimination matrices, M_1 and M_2 shown below.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 5 \\ 3 & -3 & 7 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

If $M_2 * M_1 * A = U$ what is L?

(a)

$$L = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

(c)

$$L = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

20. Estimate a root of the polynomial $f(x) = x^3 + x + 3 = 0$ by performing one step of Newton's method, beginning with $x_0 = -1$.

(a) $x_1 = -3/2$

(b) $x_1 = -5/4$

(c) $x_1 = -1$

(d) $x_1 = -3/4$

21. Given the expression,

$$x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})},$$

which of the following iterative methods is this expression equivalent to?

(a) Bisection method

(b) Newton's method

(c) Secant method

(d) Gauss-Siedel method

22. What is the product of the following vectors?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} * \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

(d) This product is not valid.

23. The LU decomposition method is computationally more efficient than the Gaussian elimination method for solving ______(fill in the blank)___.

(a) the single linear equation $Ax = b$

(b) multiple equations of the form $Ax = b$ with different coefficient matrices (A) and the same right hand side vector (b)

(c) multiple equations of the form $Ax = b$ with the same coefficient matrix (A) but different right hand side vectors (b)

(d) the single linear equation $Ax = b$ where A is singular

24. We want to compute $G(x)$ where $G(x) = 3x^2 + 2$ and $x = 1$. What is the absolute condition number of this problem?

(a) 4

(b) 5

(c) 6

(d) $+\infty$

25. An iterative solution of a non-linear equation is subject to _____(fill in the blank)___.

(a) round-off error

(b) truncation error

(c) both round-off and truncation error

(d) neither round-off nor truncation error

26. Given the code below, compute the BEST asymptotic bound for the number of floating point operations needed to compute `x` based on the value n.

```
x(1) = b(1)/L(1,1);
for i = 2:n
  s = b(i);
  for j = 1:i-1
    s = s - L(i,j)*x(j);
  end
    x(i) = s/L(i,i);
end
```

(a) $O(n)$

(b) $O(n^2)$

(c) $O(n^3)$

(d) $O(n^4)$

27. An iterative scheme to solve non-linear equations is said to have quadratic convergence if it satisfies which of the following relations where e_n is the absolute value of the error after iteration n,(A is some constant and $\sim$ means approximately equal for large n)

(a) $e_{n+1} - e_n \sim n^2$

(b) $e_{n+1}/e_n^2 \sim A$

(c) $e_{n+1} \sim A/e_n^2$

(d) $e_{n+1} \sim e_n/2$

28. Partial pivoting in Gaussian Elimination algorithm is necessary to -----(fill in the blank)---.

(a) avoid the failure of the algorithm if a pivot coefficient is equal to zero

(b) reduce the condition number

(c) reduce truncation error

(d) transpose the matrix

29. What is 2-norm of the following matrix?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(a) 1

(b) 2

(c) 3

(d) 4

30. Which of the following is **True** for an arbitrary ill-conditioned matrix A?

(a) A has a large condition number.

(b) $\det(A) = 0$.

(c) For an equation $Ax = b$, Gaussian elimination guarantees an exact solution.

(d) The ratio $\frac{\sigma_{max}}{\sigma_{min}}$ of largest to smallest singular values of A is small.

31. A matrix A is SPD (symmetric, positive definite) if and only if it has a Cholesky Factorization.

(a) True

(b) False

For the next two questions, use the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ -1 & 2 & -6 \end{bmatrix}$$

32. What is the matrix L when you factor $A = LU$ without pivoting?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & -4 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$

33. What is the matrix U when you factor $A = LU$ without pivoting?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 4 & 9 \end{bmatrix}$

34. The $n \times n$ matrix A is positive definite, that is, $x^T Ax > 0$ for any $n \times 1$ non-zero vector x. Assume further that A has the Doolittle factorization, $A = LDL^T$ where D is a diagonal matrix and L is lower triangular and non-singular. Is the following statement about A true or false?

A is positive definite if and only if all the diagonal elements of D are positive.

(a) True

(b) False

35. Which of the following is **True** concerning the solution to the normal equations $A^T Ax = A^T b$?

(a) The normal equations always has a unique solution.

(b) Normal equations has a unique solution when the matrix A has full rank.

(c) Solving the normal equations is always the best way to solve a linear least squares problem.

(d) The condition number of the matrix $A^T A$ in the normal equations is always equal to one.

36. Find a least square solution $Ax \cong b$ given the QR factorization of A as,

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = QR = \begin{bmatrix} -1/3 & -2/3 & 2/3 \\ -2/3 & -1/3 & -2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix} * \begin{bmatrix} -3 & 3 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$$

(a) $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $x = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

(c) $x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(d) $x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

37. Find the least square solution $Ax \cong b$ given the SVD factorization of A as,

$$A = USV^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

(a) $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(d) $x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

38. What is **TRUE** about SVD decomposition of A?

(a) SVD decomposition is unique.

(b) S is a square matrix.

(c) U is the matrix of eigenvectors of A^TA.

(d) V is the matrix of eigenvectors of A^TA.

39. Consider $A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Using Jacobi's iterative method to approximate the solution $Ax = b$ with a starting guess of $x_0 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, what is x_1?

(a) $\begin{bmatrix} 1/3 \\ 6/5 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2/3 \\ 4/5 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 4/3 \\ 1/5 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1/3 \\ 2/5 \\ 3 \end{bmatrix}$

40. Apply two iterations of the normalized power method to the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ using $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a starting vector. After normalization using the ∞-norm, one entry of the resulting vector will be one. What is the value of the other entry?

(a) 1/4

(b) -1/4

(c) 1/2

(d) -1/2

41. Which one of the following statements is **True**?

(a) It is always better to use higher degree interpolating polynomials.

(b) Interpolation can only be performed using polynomials.

(c) Interpolating polynomial is unique.

(d) We use least squares methods in finding interpolation polynomials.

42. Why are Chebyshev Nodes useful?

(a) They make calculation faster.

(b) They make the condition number smaller.

(c) They make interpolating error smaller.

(d) They make the residual smaller.

43. Given the two points $(x, y) = (2, 4)$ and $(x, y) = (3, 9)$, which of the following functions can be a Lagrange basis function used in interpolation through these points?

(a) $x - 2$

(b) $\dfrac{x-2}{-5}$

(c) $x - 3$

(d) $\dfrac{x-3}{5}$

44. Consider two points $(x_1, y_1) = (2, 4)$, and $(x_2, y_2) = (3, 9)$, what is an equation of a Newton form interpolating polynomial?

(a) 2

(b) $4 + 5(x - 4)$

(c) $9 + 5(x - 9)$

(d) $9 + 5(x - 3)$

45. Using high degree polynomials for interpolation always gives better accuracy.

(a) True

(b) False

46. A function $S(x)$ with domain $[a, b]$ is a spline of degree 3. What is *NOT* always the case?

(a) $S(x)$ is continuous on $[a, b]$.

(b) $S'(x)$ is continuous on $[a, b]$.

(c) $S''(x)$ is continuous on $[a, b]$.

(d) $S'''(x)$ is continuous on $[a, b]$.

47. Degree 3 Splines are unique.

(a) True

(b) False

48. Consider two points $(a, f(a)) = (3, 9)$ and $(b, f(b)) = (4, 5)$, estimate the integral $\int_a^b f(x)dx$ using the Trapezoid rule.

(a) 6.5

(b) 7

(c) 10

(d) not enough information

49. Consider three points $(a, f(a)) = (3, 9)$, $((a + b)/2,\ f((a + b)/2)) = (4, 5)$ and $(b, f(b)) = (5, 7)$, estimate the integral $\int_a^b f(x)dx$ using Simpson's 1/3 rule.

(a) 6.5

(b) 7

(c) 10

(d) 12

50. If you want to integrate $\int_a^b 3 + 4x + 9x^2\ dx$ which of the formulas below require the minimum number of function evaluations to compute the integral exactly (assuming no roundoff error)? Hint: consider the error formulas.

(a) Trapezoid

(b) Simpson's 1/3

(c) Simpson's 3/8

(d) Boole's

Extra Credit

Answering, the questions below, correctly will add points to your exam total. Answering incorrectly or not answering will not add points to your exam total.

51. We want to use a Monte Carlo method to compute the area of a region (named A) in the 2-dimensional plane. The region A can be completely enclosed by a rectangle (named R) of length equal to L and width W. If we generate N random points in the rectangle R and exactly K of those points lie in the rectangle R but not in the region A then we can estimate the area of region A by which of the following formulas?

(a) $\frac{(N-K)*L*W}{N}$

(b) $\frac{K*L*W}{N}$

(c) $\frac{(N-K)*L*W}{N+K}$

(d) $\frac{K*L*W}{N+K}$

52. Let the $n \times n$ matrix named V have columns denoted by v_i for $i = 1, \ldots, n$ and an $n \times 1$ vector named a have values a_i for $i = 1, \ldots, n$ then the sum $\sum_{i=1}^{n} a_i * v_i$ can be written in which of the following equivalent way?

(a) $V^T a^T$

(b) aV

(c) $a^T V^T$

(d) Va