CS357 **Spring 2011**
Final Exam (2 Hours) **Monday, May 9th**

No computers, calculators, or books allowed.
Choose closest solution.
You may take with you two "cheat sheets" - two sheets of size 8.5" x 11" or smaller.

---

1. **True or False** A permutation matrix merely reorders the components of a vector and does not change their values, so it preserves the vector 1-norm.

2. **True or False** A shortcoming of normal equations to solve the least square problems is that sensitivity of solution will worsen since $\text{cond}(\mathbf{A}^T\mathbf{A}) = [\text{cond}(\mathbf{A})]^2$

3. **True or False** The inverse of a tri-diagonal matrix is also tri-diagonal.

4. **True or False** If $\mathbf{A}$ is symmetric and positive definite, then Gaussian elimination without pivoting applied to the linear system $\mathbf{Ax} = \mathbf{b}$ is backwards stable.

5. **True or False** Initial value Ordinary Differential Equation problems always have unique solution

6. **True or False** For a **continuous** function on a fixed interval, the interpolating polynomial based on equally spaced points always converges to the function as the number of interpolating points increases.

7. **True or False** The following loop terminates when $x = 1.0$.

```
x = 0.0;
while x ~= 1.0
  x = x + 0.1;
  disp(x)
end
```

8. The product of the interval "numbers" $[-2, 3]*[4, 5]$ is

   (a) $[8, 15]$
   (b) $[-10, 15]$
   (c) $[-12, 15]$
   (d) $[-2, 5]$

9. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$ is

(a) a Bijection

(b) a Injection

(c) a Surjection

(d) none of the above

The following are "mathematically equivalent" expressions for $x \in [-1, 1)$:

(i) $\sqrt{1+x} - \sqrt{1-x}$

(ii) $(\sqrt{1+x}/\sqrt{1-x} - 1)(\sqrt{1-x})$

(iii) $\dfrac{2x}{\sqrt{1+x} + \sqrt{1-x}}$

10. Which of the following is true under IEEE double precision arithmetic?

(a) For x near 0, expression (i) will be typically more accurate.

(b) For x near 0, expression (ii) will be typically more accurate.

(c) For x near 0, expression (iii) will be typically more accurate.

(d) All three will produce same result for $|x| < \epsilon_{\text{mach}}$

11. Let $F(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function and $J_F|_{\mathbf{x}}$ be its Jacobian evaluated at the point $\mathbf{x} \in \mathbb{R}^n$. Which of the following gives the the first two terms of the Taylor series expansion of $F(\mathbf{y})$ about the point $\mathbf{x}$?

(a) $F(\mathbf{y}) \approx F(\mathbf{x}) - J_F|_{\mathbf{x}}(\mathbf{y} - \mathbf{x})$

(b) $F(\mathbf{y}) \approx F(\mathbf{x}) + J_F|_{\mathbf{y}}(\mathbf{y} - \mathbf{x})$

(c) $F(\mathbf{y}) \approx F(\mathbf{x}) + J_F|_{\mathbf{x}}(\mathbf{y} - \mathbf{x})$

(d) $F(\mathbf{y}) \approx F(\mathbf{x}) + \frac{1}{2} J_F|_{\mathbf{x}}(\mathbf{y} - \mathbf{x})$

12. We approximate an infinitely differentiable function $f(x) : \mathbb{R} \to \mathbb{R}$ with a Taylor series until order 4 (5 terms) at $x = 0$. What is the maximum error for x near zero in Big O notation?

(a) $\mathcal{O}(x^4)$

(b) $\mathcal{O}(x^5)$

(c) $\mathcal{O}(x^6)$

(d) $\mathcal{O}(x^{10})$

13. The iterative solution using Newton's method to a find root of a system of non-linear equations is subject to

(a) round-off error

(b) truncation error

(c) none of these

(d) both 1 and 2

14. Consider the function

$$f(\mathbf{x}) = \begin{bmatrix} 3x_1^2 + 2x_2 \\ x_2 + 1 \end{bmatrix}.$$

Apply one iteration of Newton's method with initial guess $\mathbf{x}_0 = [1, 2]^\mathsf{T}$. What is $\mathbf{x}_1$?

(a) $[\frac{7}{6}, 5]^\mathsf{T}$

(b) $[\frac{5}{6}, -1]^\mathsf{T}$

(c) $[\frac{7}{6}, -1]^\mathsf{T}$

(d) $[\sqrt{\frac{2}{3}}, -1]^\mathsf{T}$

15. Consider the function

$$f(x) = 1 - 3x^2$$

Apply one iteration of the secant method with initial guesses $x_0 = 0$ and $x_1 = 1$ What is x_2?

(a) $-\frac{1}{3}$

(b) 0

(c) $\frac{1}{3}$

(d) $\frac{5}{3}$

16. Which root finding method is the following algorithm representing
1 initialize: $x_1 = ...$
2 for k = 2, 3, ...
3 $x_k = x_{k-1} - f(x_{k-1})/f'(x_{k-1})$
4 if converged, stop
5 end

(a) Bisection method

(b) Secants method

(c) Newton's method

17. What is one advantage of the bisection method over the secant method?

- **(a)** Assuming the initial interval brackets a root, the bisection method guaranteed to converge.
- **(b)** The bisection method has faster asymptotic convergence.
- **(c)** After the first iteration, the bisection method requires fewer new function evaluations.
- **(d)** Bisection method has no advantage over secant method

18. Applying the Secant method to $f(x) = (x-1)^2$ with initial guess $x_0 = 0$, what is the value of x after two iterations?

- **(a)** 1/2
- **(b)** 3/4
- **(c)** 1
- **(d)** Insufficient data to solve using secant method

19. When applying Newton's method to find a solution to the following system of nonlinear equations $x_1 * x_2 = 0$ and $2x_1 + x_2 = 1$ with the starting value $[x_1 \ \ x_2]^T = [1 \ 1]^T$, what is the result of a single iteration?

- **(a)** $[1/2 \ 0]^T$
- **(b)** $[0 \ 1]^T$
- **(c)** $[1 \ 1]^T$
- **(d)** $[0.5 \ 0.5]^T$

20. Assume that we use a new root finding method. For this method, at the n^{th} step, the error is $1/n$. What is the convergence rate?

- **(a)** Cubic
- **(b)** Super linear
- **(c)** Quadratic
- **(d)** Sublinear

21. LU decomposition reduces solving a general linear system to solving two __________ systems.

- **(a)** Diagonal
- **(b)** Triangular
- **(c)** Orthogonal
- **(d)** Symmetric

22. What is the rank of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) 0
(b) 1
(c) 2
(d) 3

23. Which of the following is the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

(a) $\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix}$

(b) $\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix}$

(c) $\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 1 & 0 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix}$

(d) $\mathbf{A}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 2 \\ -2 & 3 & -1 \end{bmatrix}$

24. Let $\mathbf{PA} = \mathbf{LU}$ be the LUP-factorization of $\mathbf{A}$. Which of the following is always true?

(a) $\det \mathbf{A} = \det \mathbf{L}$
(b) $\det \mathbf{A} = \det \mathbf{U}$
(c) $\det \mathbf{A} = \pm \det \mathbf{L}$, where the sign depends on the matrix $\mathbf{P}$
(d) $\det \mathbf{A} = \pm \det \mathbf{U}$, where the sign depends on the matrix $\mathbf{P}$

The next two questions concern the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

25. Let $\mathbf{A} = \mathbf{LU}$ be the LU-factorization of $\mathbf{A}$. What is $\mathbf{L}$?

(a) $\mathbf{L} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

(b) $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

(c) $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

(d) $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

26. Let $\mathbf{A} = \mathbf{LU}$ be the LU-factorization of $\mathbf{A}$. What is $\mathbf{U}$?

(a) $\mathbf{U} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{U} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\mathbf{U} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\mathbf{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Use the following matrix $\mathbf{A}$ for the next 3 questions,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$

27. What is the 2-norm of the matrix $\mathbf{A}$?

- **(a)** $\sqrt{2}$
- **(b)** $2\sqrt{2}$
- **(c)** $3\sqrt{2}$
- **(d)** 2

28. What is the 2-norm condition number of the matrix $\mathbf{A}$?

- **(a)** $\sqrt{2}$
- **(b)** $2\sqrt{2}$
- **(c)** $3\sqrt{2}$
- **(d)** 2

29. What are the singular values of the matrix $\mathbf{A}$.

- **(a)** 2 and 8
- **(b)** 10 and 16
- **(c)** $\sqrt{2}$ and $2\sqrt{2}$
- **(d)** $\sqrt{2}$ and $3\sqrt{2}$

30. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

If partial pivoting is used, what is the **value** of the first pivot element?

- **(a)** 1
- **(b)** 7
- **(c)** 13
- **(d)** 16

31. Which of the following is true?

(a) The QR factorization of a matrix is unique

(b) The singular value decomposition of a matrix is unique

(c) The solution of linear least squares problem is unique

(d) If the null space of matrix A is zero, then the normal equations of the least square problem have a unique solution

32. Consider $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$. Using Jacobi's iterative method to approximate the solution $\mathbf{Ax} = \mathbf{b}$ with a starting guess of

$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, what is $\mathbf{x}_1$?

(a) $\begin{bmatrix} 2 \\ 4/3 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 3/2 \end{bmatrix}$

(d) $\begin{bmatrix} 1/3 \\ 0 \\ 2 \end{bmatrix}$

33. If a first-degree polynomial of the form $y = x_1 + x_2 t$ is fit to the three data points $(t_i, y_i) = (1, 1), (2, 1), (3, 2)$ by linear least squares, what is the least squares solution?

(a) $x_1 = 1/3$, $x_2 = 1/2$

(b) $x_1 = 1$, $x_2 = 0$

(c) $x_1 = -1$, $x_2 = 1$

(d) $x_1 = 1/2$, $x_2 = 1/2$

34. Which matrix has the CSR representation

$$\begin{aligned} AA = & \ [1\ \ 2\ \ 4\ \ 3\ \ 5] \\ JA = & \ [1\ \ 1\ \ 2\ \ 1\ \ 3] \\ IA = & \ \ \ [1\ \ 2\ \ 4\ \ 6] \end{aligned}$$

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 0 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$

35. Apply two iterations of normalized power method to the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ using $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as starting vector. After normalization using the ∞-norm, one entry of the resulting vector will be one. What is the value of the other entry?

(a) 7/13

(b) 8/13

(c) 9/13

(d) 10/13

36. Consider Compressed Sparse Row (CSR) of the matrix $\mathbf{A} = \begin{bmatrix} 4 & 0 & 6 \\ 0 & 7 & 9 \\ 8 & 0 & 10 \end{bmatrix}$, what is IA?

(a) [1 3 5 7]

(b) [2 3 4 5]

(c) [3 4 5 6]

(d) [5 6 7 7]

37. Consider the following Matlab code

```
i = [2 5 3 4 4 5 1 3];

j = [1 1 2 2 3 3 5 5];

A = [1 2 3 4 5 6 7 8];

AA = sparse(i, j, A);

full(AA);
```

What is AA?

(a) $\begin{bmatrix} 0 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 8 \\ 0 & 4 & 0 & 0 & 5 \\ 2 & 0 & 6 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 8 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 8 \\ 0 & 4 & 5 & 0 & 0 \\ 2 & 0 & 6 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 8 & 0 & 0 \end{bmatrix}$

38. When interpolating a distinct data set of 7 values, the degree of a Lagrange basis function used in interpolation is

(a) 4
(b) 5
(c) 6
(d) 7

39. The Newton form of the quadratic interpolant of $f(x) = \frac{12}{2+x}$ using $x = 0, 1, 2$ is

(a) $p_2(x) = 6 - 2x + 1/2x(x-1)$

(b) $p_2(x) = 6 - 4x + 3x(x-1)$

(c) $p_2(x) = 6 - 2x + 3/2x(x-1)$

(d) $p_2(x) = 6x(x-1) - 2x + 1/2$

40. The computational cost of evaluating a polynomial of degree n using Horner's method is

(a) $O(n^{1/2})$

(b) $O(n)$

(c) $O(n^2)$

(d) $O(n^3)$

41. Approximate $\int_0^2 f(x)dx$ using Trapezoid rule, given that $f(0) = 1$ and $f(2) = 6$.

(a) 3.5

(b) 7

(c) 4

(d) 5

42. Approximate $\int_0^2 f(x)dx$ using Simpson's rule, given that $f(0) = 1$, $f(1) = 2$ and $f(2) = 6$.

(a) 3.5

(b) 7

(c) 4

(d) 5

43. Approximate the integral $\int_{-1}^1 x^3 \, dx$ using two point Gaussian quadrature. (the Gaussian nodes are $\pm\frac{1}{\sqrt{3}}$ and the corresponding weights are both 1)

(a) $2\left(\frac{\sqrt{3}}{3}\right)^3$

(b) $-2\left(\frac{\sqrt{3}}{3}\right)^3$

(c) 0

(d) $\frac{2}{\sqrt{3}}$

44. Which of the following has the best error bound if the intervals are small enough?

(a) Composite Trapezoid rule

(b) Composite Simpson's 1/3 rule

(c) Composite Simpson's 3/8 rule

(d) Composite Boole's rule

45. Which of the following values of k makes $y = 1000e^{kt}$ go to zero if $t \to \infty$.

(a) 2

(b) 1

(c) 0

(d) -1

46. What is the value of the first step z_1 of the forward Euler method for the following problem with initial condition $z_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and time step 0.1? $z' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} z$

(a) $\begin{bmatrix} 0.4 \\ 0 \\ 0.3 \end{bmatrix}$

(b) $\begin{bmatrix} 0.3 \\ 0 \\ 0.2 \end{bmatrix}$

(c) $\begin{bmatrix} 1.39 \\ 0 \\ 1.29 \end{bmatrix}$

(d) $\begin{bmatrix} 1.3 \\ 0 \\ 1.2 \end{bmatrix}$

47. Consider $\frac{d\mathbf{y}(t)}{dt} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} * \mathbf{y}(t)$, what is the Taylor series for $\mathbf{y}$ around $t_0 = 0$ where $\mathbf{y}(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$?

(a) $\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T + t \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^T + t^2 \begin{bmatrix} 4.5 & 4.5 & 4.5 \end{bmatrix}^T + \cdots$

(b) $\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T + t \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^T + t^2 \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^T + \cdots$

(c) $\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T + t \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T + t^2 \begin{bmatrix} 1.5 & 1.5 & 1.5 \end{bmatrix}^T + \cdots$

(d) $\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T + t^2 \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^T + t^4 \begin{bmatrix} 4 & 3 & 1 \end{bmatrix}^T + \cdots$

48. Let $B = \begin{bmatrix} 0 & 0 & 0.1 \\ 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$, which one is closest to e^B?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 0.1 \\ 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0.1 \\ 0.1 & 1 & 0 \\ 0 & 0.1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}$

49. Consider $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and the diagonalization of A is $D = S^{-1}\mathbf{A}S$, what can you say about the eigenvalues of the matrix A?

(a) The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of $D^T D$.

(b) The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of S.

(c) The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of S^{-1}.

(d) The eigenvalues of $\mathbf{A}$ are the same as the eigenvalues of D.

50. Consider

$$\begin{aligned} x' &= xy + 2 \\ y' &= z + x \\ z' &= x + y + z \end{aligned}$$

If $[x_0\ y_0\ z_0]^T = [1\ 1\ 1]^T$ and time step is 0.1 what is the next step of forward Euler?

(a) $[1.1\ 1.1\ 1.1]^T$

(b) $[0.7\ 0.8\ 0.7]^T$

(c) $[1.3\ 1.2\ 1.3]^T$

(d) $[0.3\ 0.2\ 0.3]^T$