Optimization (Introduction)

Optimization

Goal: Find the **minimizer** $\boldsymbol{x}^*$ that minimizes the **objective (cost) function** $f(\boldsymbol{x}): \mathcal{R}^n \rightarrow \mathcal{R}$

Unconstrained Optimization

Optimization

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Constrained Optimization

Unconstrained Optimization

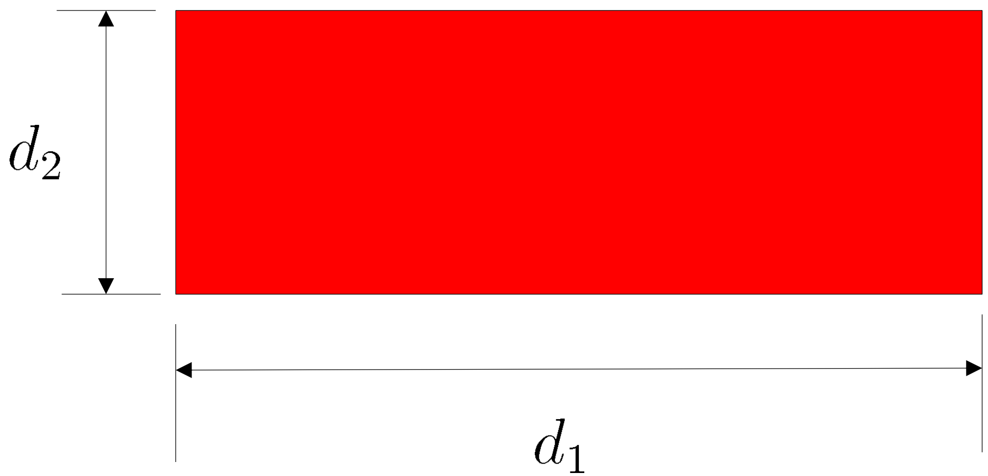
- **What if we are looking for a maximizer $\boldsymbol{x}^*$?**

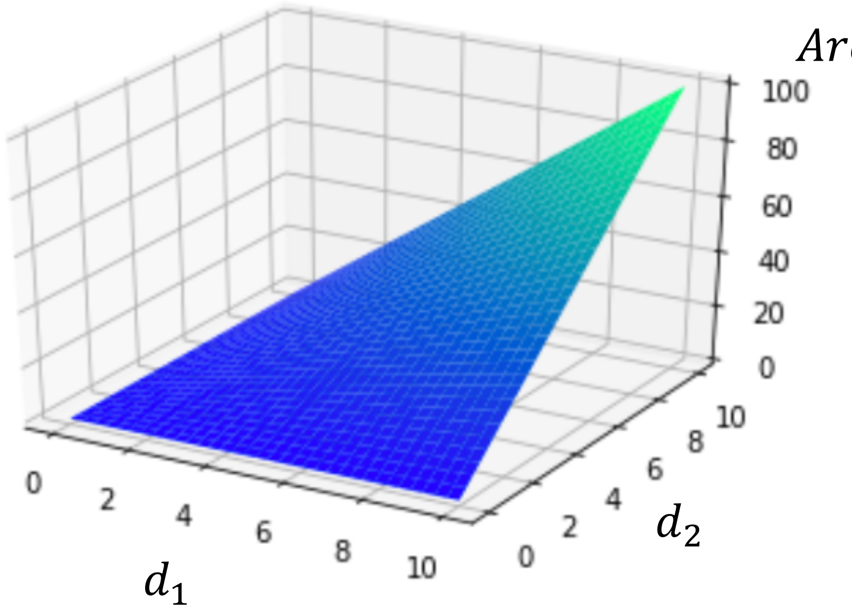
$$f(\boldsymbol{x}^*) = \max_{\boldsymbol{x}} f(\boldsymbol{x})$$

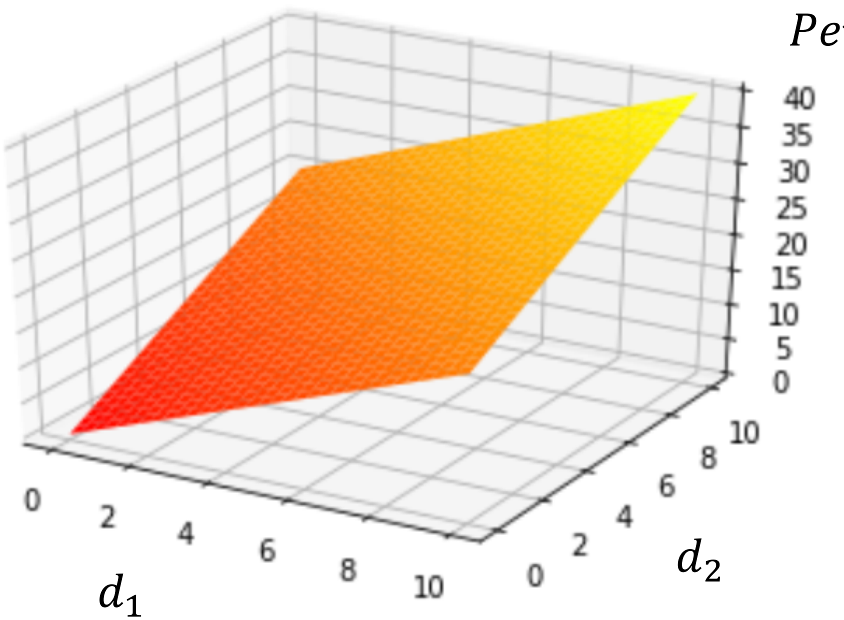
Calculus problem: maximize the rectangle area subject to perimeter constraint

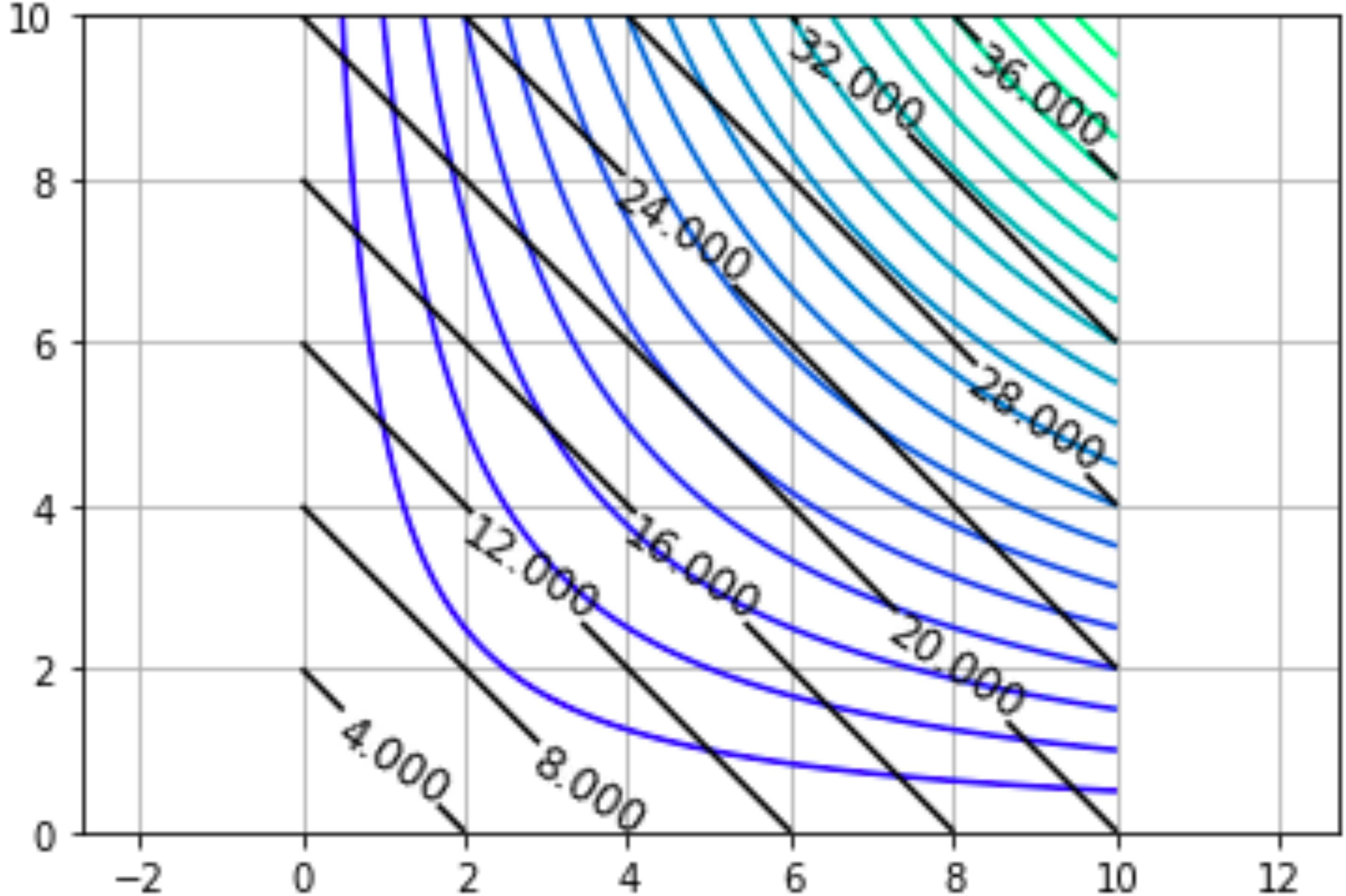
$$\max_{\boldsymbol{d} \in \mathcal{R}^2} \quad f(d_1, d_2) = d_1 \times d_2$$

$$\text{such that} \quad g(d_1, d_2) = 2\,(d_1 + d_2) - 20 \leq 0$$




$Area = d_1 d_2$
100
80
60
40
20
0
0
2
4
6
8
10
d_1
0
2
4
6
8
10
d_2


$Perimeter = 2(d_1 + d_2)$
40
35
30
25
20
15
10
5
0
0
2
4
6
8
10
d_1
0
2
4
6
8
10
d_2



10
8
6
4
2
0
−2
0
2
4
6
8
10
12
4.000
8.000
12.000
16.000
20.000
24.000
28.000
32.000
36.000

Unconstrained Optimization 1D

What is the optimal solution? (1D)

$$f(x^*) = \min_x f(x)$$

(First-order) Necessary condition

(Second-order) Sufficient condition

Types of optimization problems

$$f(x^*) = \min_x f(x)$$

f: nonlinear, continuous and smooth

Gradient-free methods

Evaluate $f(x)$

Gradient (first-derivative) methods

Evaluate $f(x), f'(x)$

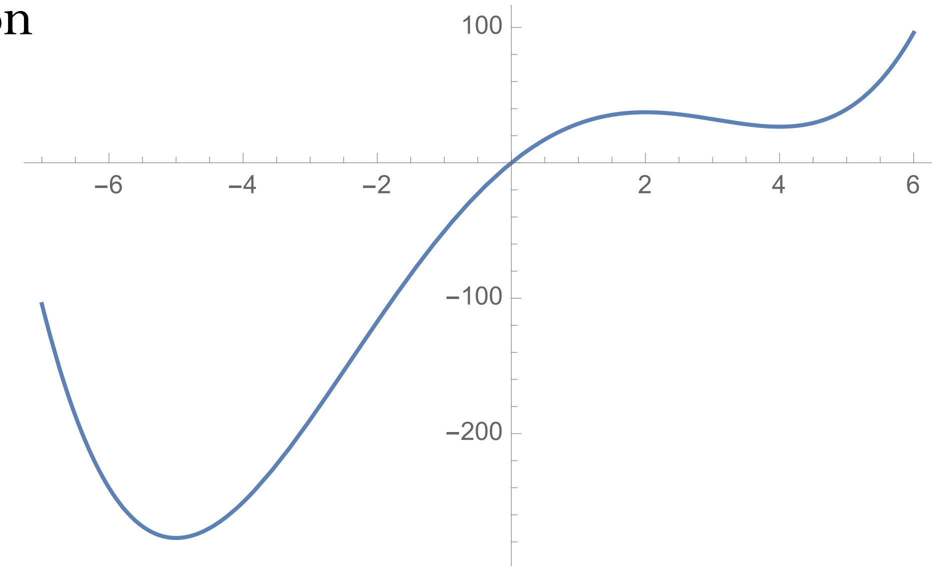
Second-derivative methods

Evaluate $f(x), f'(x), f''(x)$

Does the solution exists? Local or global solution?

Example (1D)

Consider the function $f(\boldsymbol{x}) = \frac{x^4}{4} - \frac{x^3}{3} - 11\,x^2 + 40x$. Find the stationary point and check the sufficient condition



Optimization in 1D: Golden Section Search

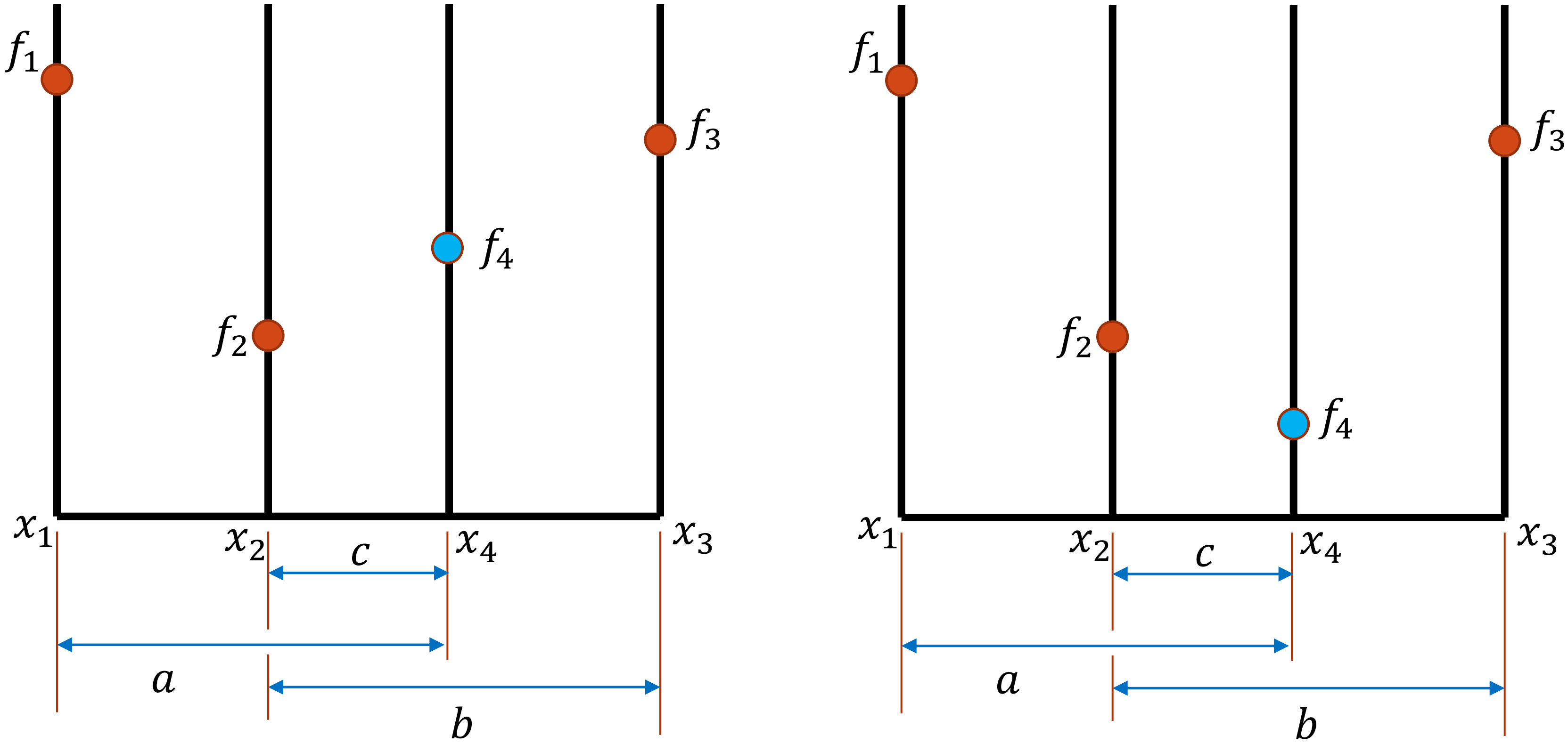
- Similar idea of bisection method for root finding
- Needs to bracket the minimum inside an interval
- Required the function to be unimodal

A function $f: \mathcal{R} \rightarrow \mathcal{R}$ is unimodal on an interval $[a, b]$

✓ There is a unique $\boldsymbol{x}^* \in [a, b]$ such that $f(\boldsymbol{x}^*)$ is the minimum in $[a, b]$

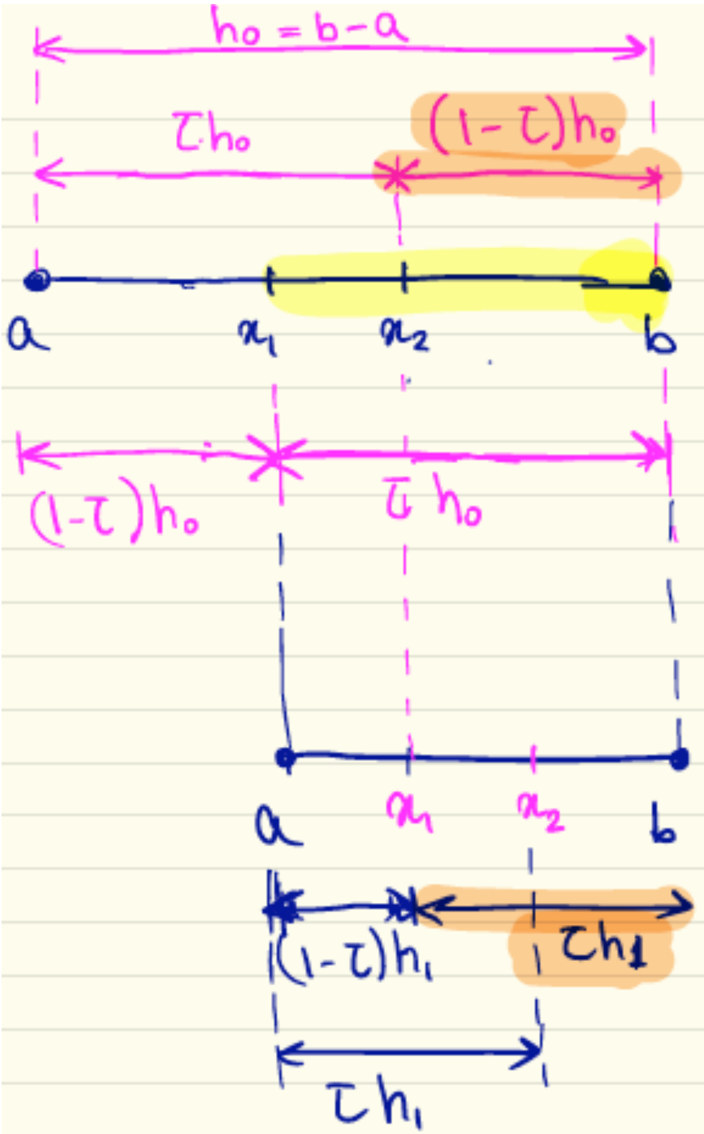
✓ For any $x_1, x_2 \in [a, b]$ with $x_1 < x_2$

- $x_2 < \boldsymbol{x}^* \Longrightarrow f(x_1) > f(x_2)$
- $x_1 > \boldsymbol{x}^* \Longrightarrow f(x_1) < f(x_2)$



f_1
f_2
f_3
f_4
x_1
x_2
x_4
x_3
c
a
b
f_1
f_2
f_3
f_4
x_1
x_2
x_4
x_3
c
a
b

Golden Section Search



Propose:

$$x_1 = a + (1-\tau)h_0$$

$$x_2 = a + \tau h_0$$

Evaluate $f_1 = f(x_1)$

$f_2 = f(x_2)$

if $(f_1 > f_2)$:

$a = x_1$

$x_1 = x_2$ → already have func. value!

$h_1 = b - a$

$x_2 = a + \tau h_1$

$f_2 = f(x_2)$ → only one

if $(f_1 < f_2)$:

$b = x_2$

$x_2 = x_1$

$x_1 = a + (1-\tau)h_1$

$f_1 = f(x_1)$

Golden Section Search

What happens with the length of the interval after one iteration?

$$h_1 = \tau\, h_o$$

Or in general: $h_{k+1} = \tau\, h_k$

Hence the interval gets reduced by $\boldsymbol{\tau}$

(for bisection method to solve nonlinear equations, τ=0.5)

For recursion:

$$\tau\, h_1 = (1 - \tau)\, h_o$$
$$\tau\, \tau\, h_o\ = (1 - \tau)\, h_o$$
$$\tau^2 = (1 - \tau)$$
$$\boldsymbol{\tau = 0.618}$$

Golden Section Search

- Derivative free method!
- Slow convergence:

$$\lim_{k\to\infty}\frac{|e_{k+1}|}{|e_k|} = 0.618 \quad r = 1\ (linear\ convergence)$$

- Only one function evaluation per iteration

Example

Consider running golden section search on a function that is unimodal. If golden section search is started with an initial brakcet of $[-10, 10]$, what is the length of the new bracket after 1 iteration?

A) 20
B) 10
C) 12.36
D) 7.64

Newton's Method

Using Taylor Expansion, we can approximate the function f with a quadratic function about x_0

$$f(x) \approx f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2} f''(x_0)\,(x - x_0)^2$$

And we want to find the minimum of the quadratic function using the first-order necessary condition

Newton's Method

- **Algorithm:**

$x_0 = \text{starting guess}$

$x_{k+1} = x_k - f'(x_k)/f''(x_k)$

- **Convergence:**
  - Typical quadratic convergence
  - Local convergence (start guess close to solution)
  - May fail to converge, or converge to a maximum or point of inflection

Newton's Method (Graphical Representation)

Example

Consider the function $f(x) = 4\,x^3 + 2\,x^2 + 5\,x + 40$

If we use the initial guess $x_0 = 2$, what would be the value of x after one iteration of the Newton's method?