Arrays: computing with many numbers

Some perspective

- We have so far (mostly) looked at what we can do with single numbers (and functions that return single numbers).
- Things can get much more interesting once we allow not just one, but many numbers together.
- It is natural to view an array of numbers as one object with its own rules.
- The simplest such set of rules is that of a **vector.**

Vectors

A vector is an element of a Vector Space

$$n\text{-vector:} \qquad \boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = [x_1 \quad x_2 \cdots x_n]^T$$

Vector space $\mathcal{V}$:

A vector space is a set $\mathcal{V}$ of vectors and a field $\mathcal{F}$ of scalars with two operations:

1) addition: $u + v \in \mathcal{V}$, and $u, v \in \mathcal{V}$

2) multiplication : $\alpha \cdot u \in \mathcal{V}$, and $u \in \mathcal{V}$, $\alpha \in \mathcal{F}$

Vector Space

The addition and multiplication operations must satisfy:

(for $\alpha, \beta \in \mathcal{F}$ and $u,v \in \mathcal{V}$)

Associativity: $u + (v + w) = (u + v) + w$

Commutativity: $u + v = v + u$

Additive identity: $v + 0 = v$

Additive inverse: $v + (-v) = 0$

Associativity wrt scalar multiplication: $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) \cdot v$

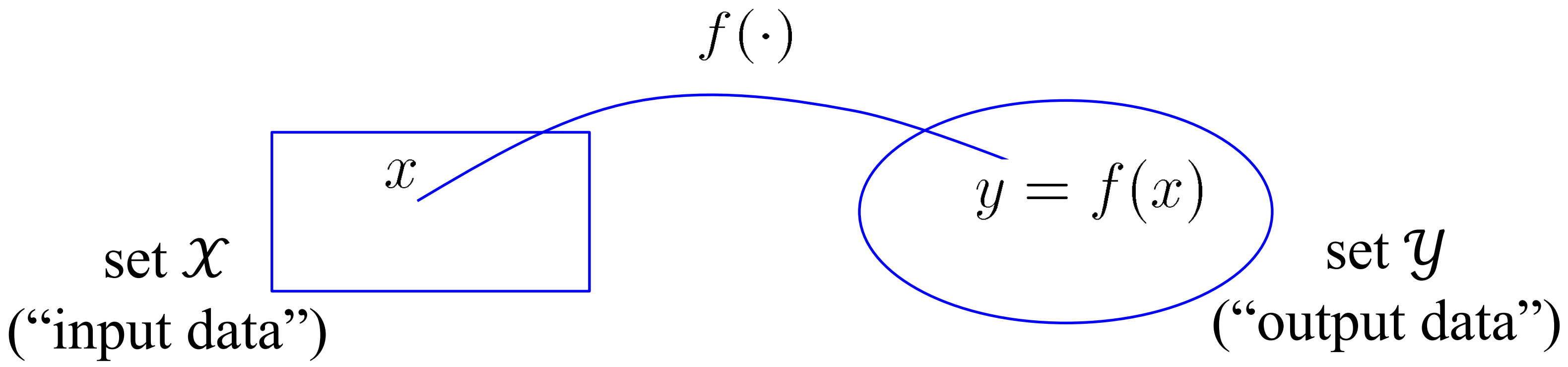
Distributive wrt scalar addition: $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$

Distributive wrt vector addition: $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$

Scalar multiplication identity: $1 \cdot (u) = u$

Linear Functions

Function: $f : \mathcal{X} \rightarrow \mathcal{Y}$



The function f takes vectors $\boldsymbol{x} \in \mathcal{X}$ and transforms into vectors $\boldsymbol{y} \in \mathcal{Y}$

A function f is a linear function if

(1) $f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$

(2) $f(a\boldsymbol{u}) = a\, f(\boldsymbol{u})$ for any scalar a

Linear functions?

$$f(x) = \frac{|x|}{x}, \ f: \mathcal{R} \to \mathcal{R}$$

$$f(x) = a\,x + b, \ f: \mathcal{R} \to \mathcal{R}, \ \ a, b \in \mathcal{R} \text{ and } a, b \neq 0$$

Matrices

- $m{\times}n$-matrix $\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{pmatrix}$

- Linear functions $f(\boldsymbol{x})$ can be represented by a Matrix-Vector multiplication.

- Think of a matrix $\boldsymbol{A}$ as a linear function that takes vectors $\boldsymbol{x}$ and transforms them into vectors $\boldsymbol{y}$

$$\boldsymbol{y} = f(\boldsymbol{x}) \;\rightarrow \boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$$

- Hence we have:

$$\boldsymbol{A}\,(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{A}\,\boldsymbol{u} + \boldsymbol{A}\,\boldsymbol{v}$$
$$\boldsymbol{A}\,(\alpha\,\boldsymbol{u}) = \alpha\,\boldsymbol{A}\,\boldsymbol{u}$$

Matrix-Vector multiplication

- Recall summation notation for matrix-vector multiplication $\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$

$$y_i = \sum_{j=1}^{n} A_{ij} x_j \qquad i = 1,2,\ldots,m$$

- You can think about matrix-vector multiplication as:

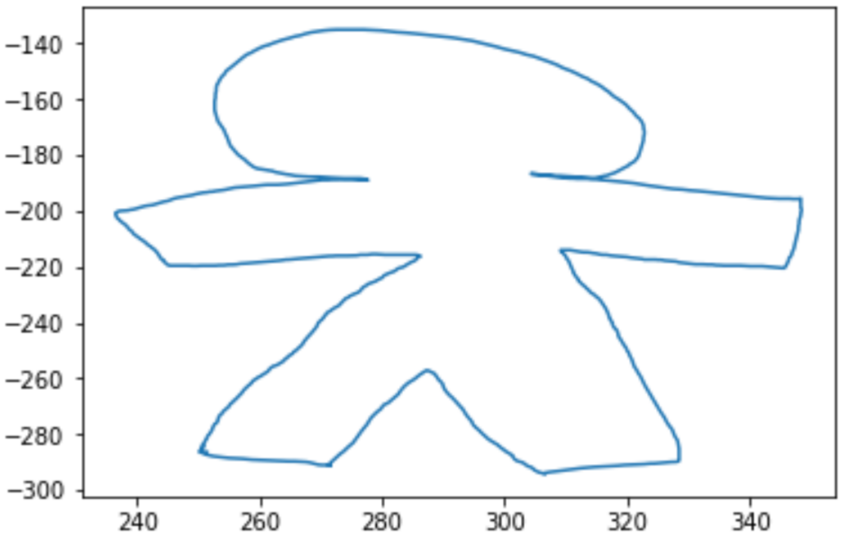
Linear combination of column vectors of **A**

$$\boldsymbol{y} = x_1\,\mathbf{A}[:,1] + x_2\,\mathbf{A}[:,2] + \cdots + x_n\,\mathbf{A}[:,n]$$

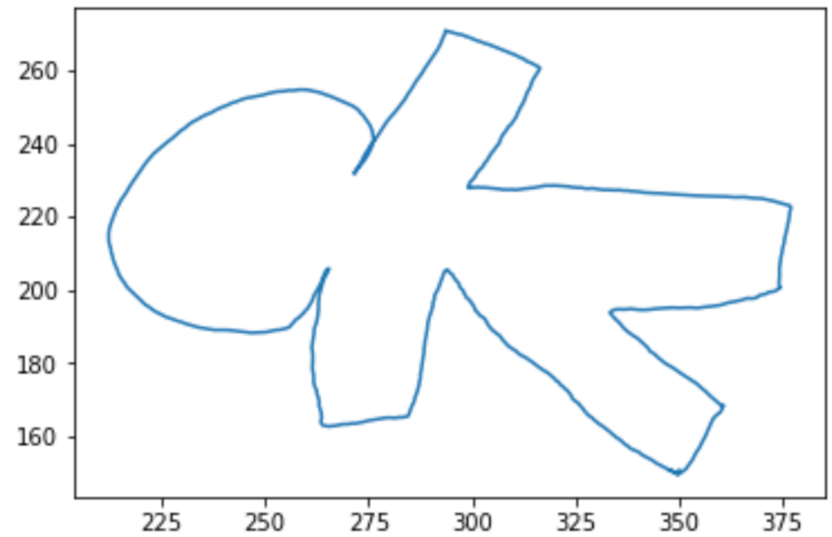
Dot product of $\boldsymbol{x}$ with rows of **A**

$$\boldsymbol{y} = \begin{pmatrix} \mathbf{A}[1,:]\cdot\boldsymbol{x} \\ \vdots \\ \mathbf{A}[m,:]\cdot\boldsymbol{x} \end{pmatrix}$$

Matrices operating on data



Data set: $\boldsymbol{x}$



Data set: $\boldsymbol{y}$

Rotation

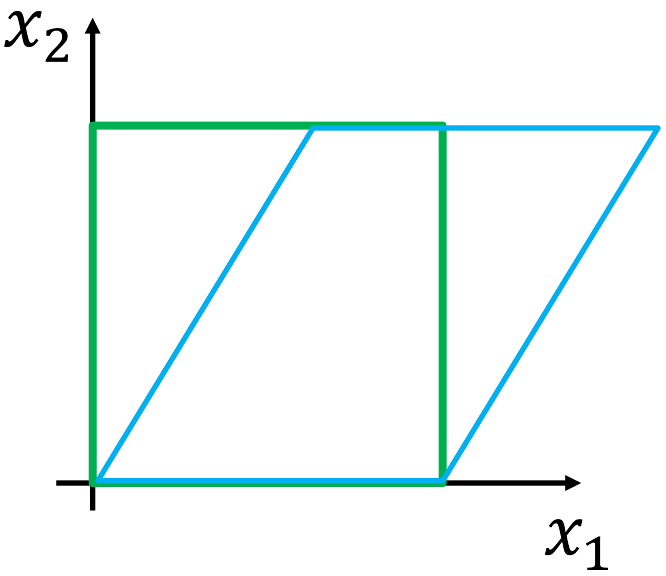
$$\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$$

or

$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x}$$

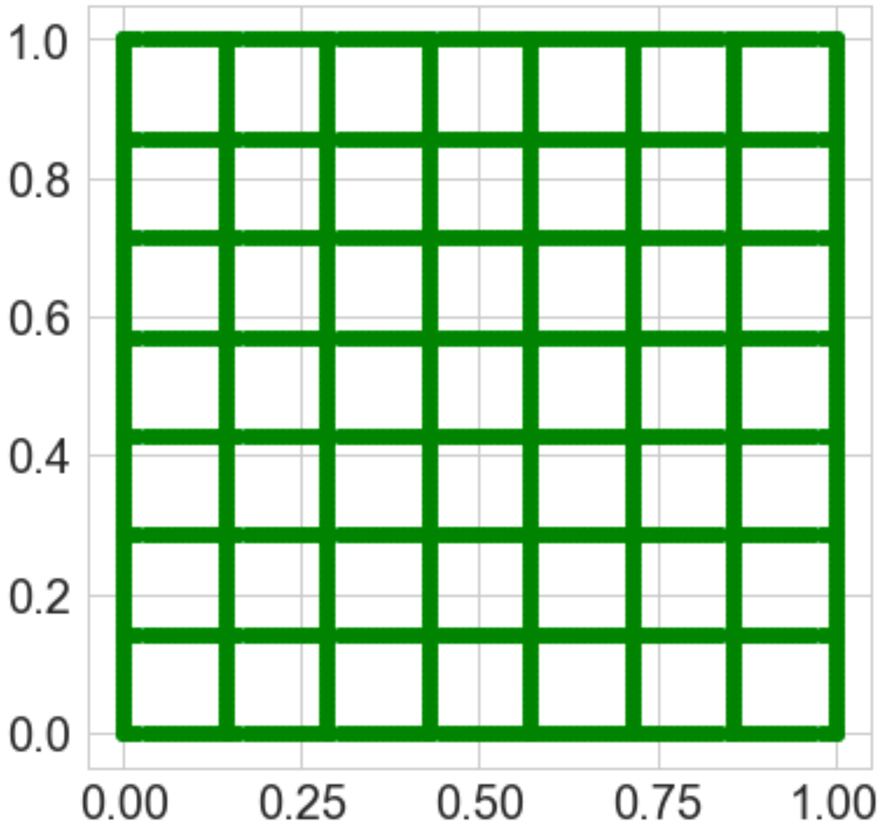
Example: Shear operator

Matrix-vector multiplication for each vector (point representation in 2D):



Matrices as operators

- **Data**: grid of 2D points
- Transform the data using matrix multiply



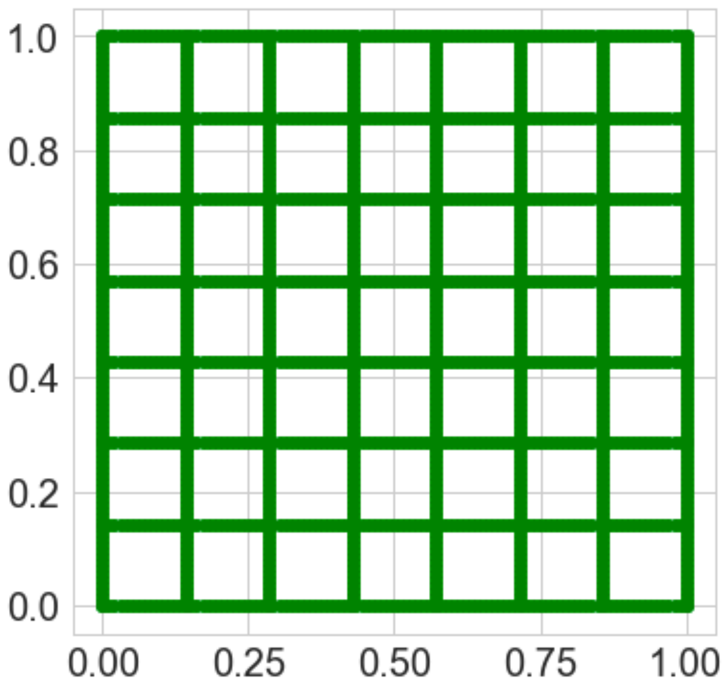
What can matrices do?

1. Shear
2. Rotate
3. Scale
4. Reflect

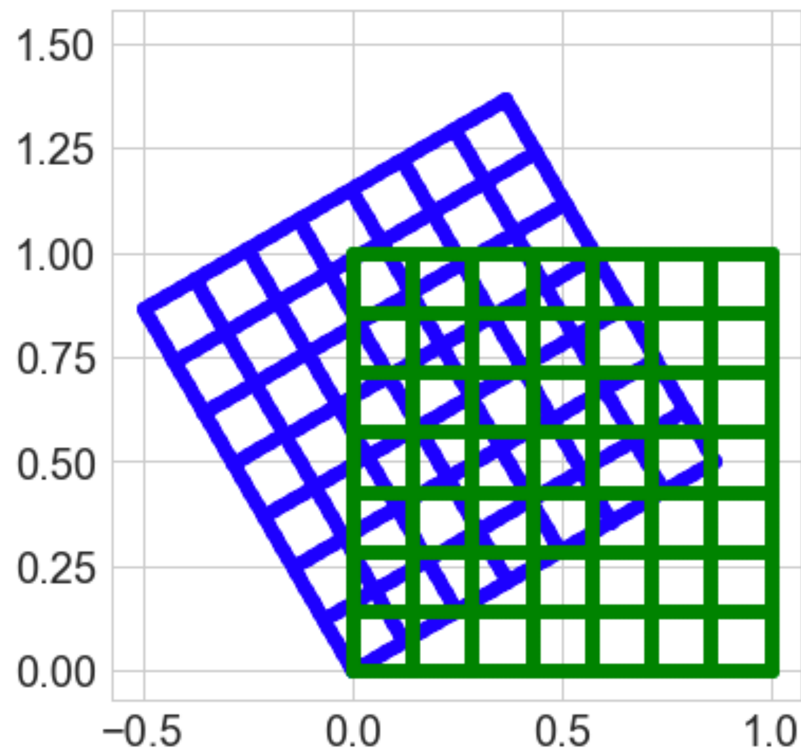
5. Can they translate?

Rotation operator

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

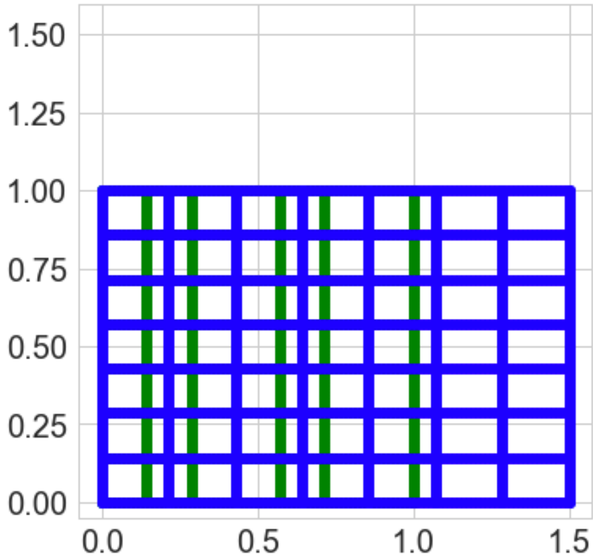


$\theta = \pi/6$



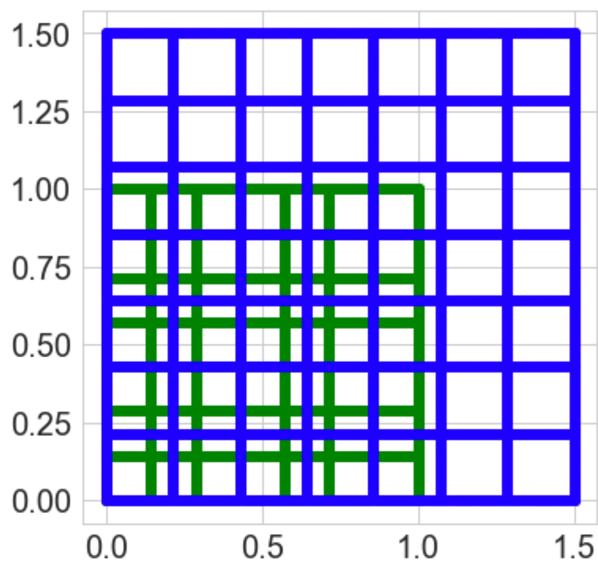
Scale operator

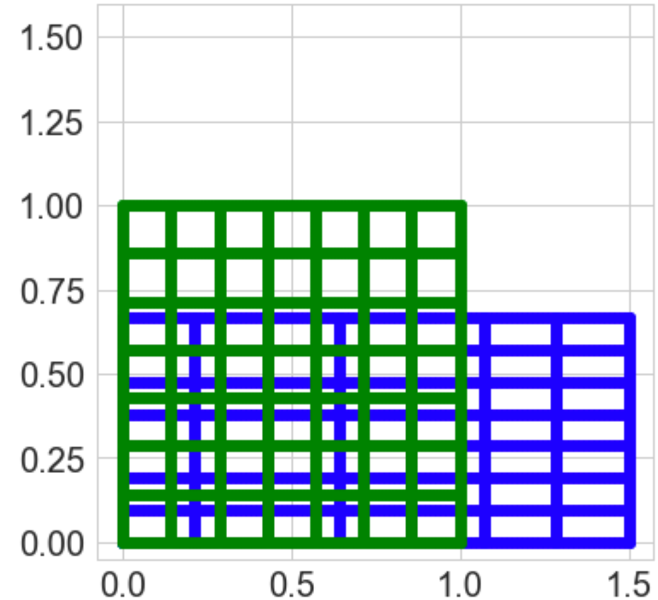
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$$\begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$



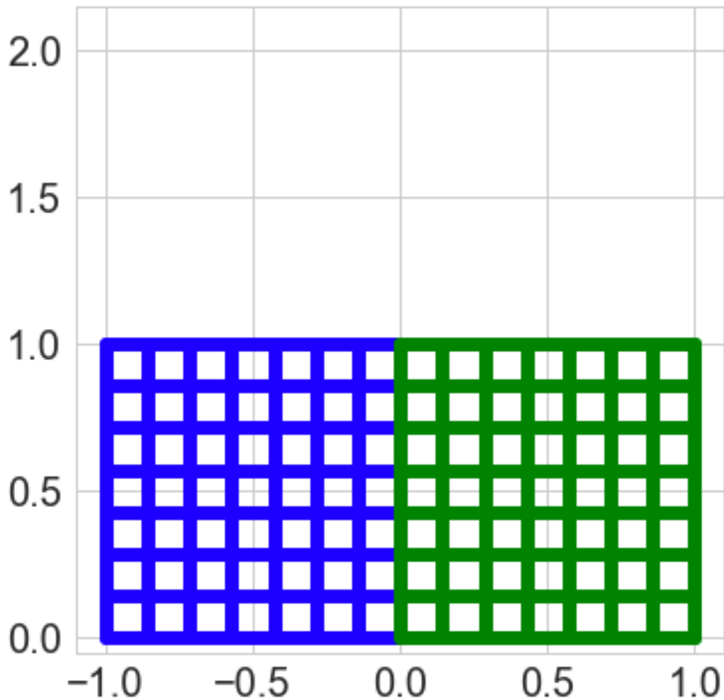


$$\begin{pmatrix} 3/2 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Reflection operator

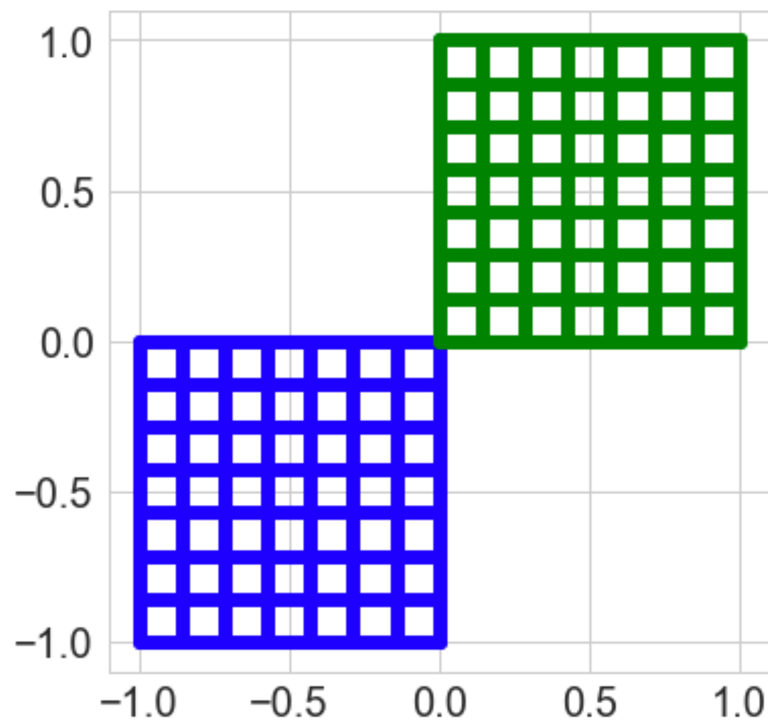
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Reflect about y-axis

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

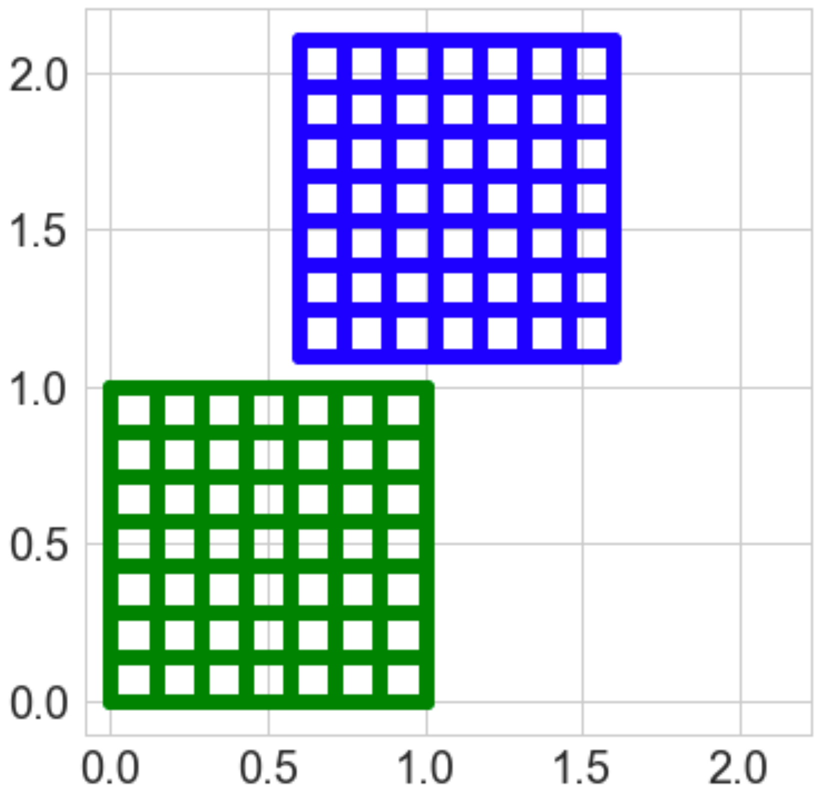


Reflect about x and y-axis

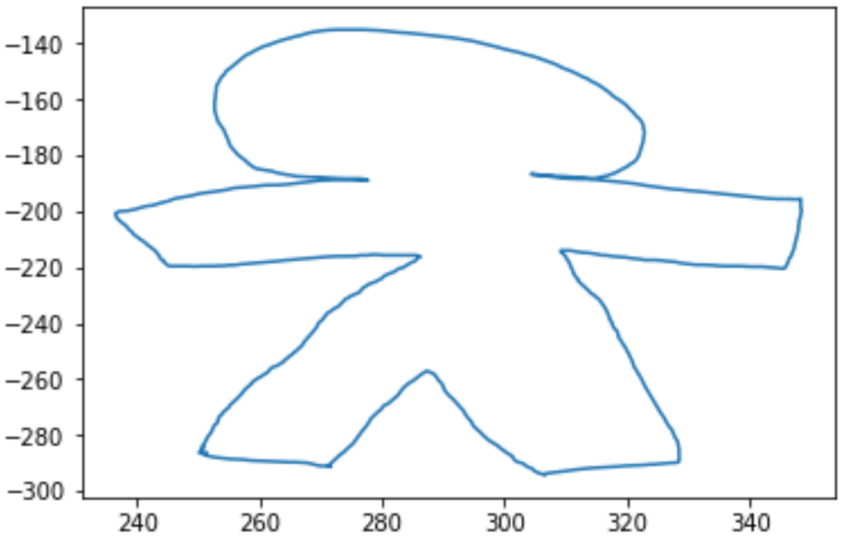
Translation (shift)

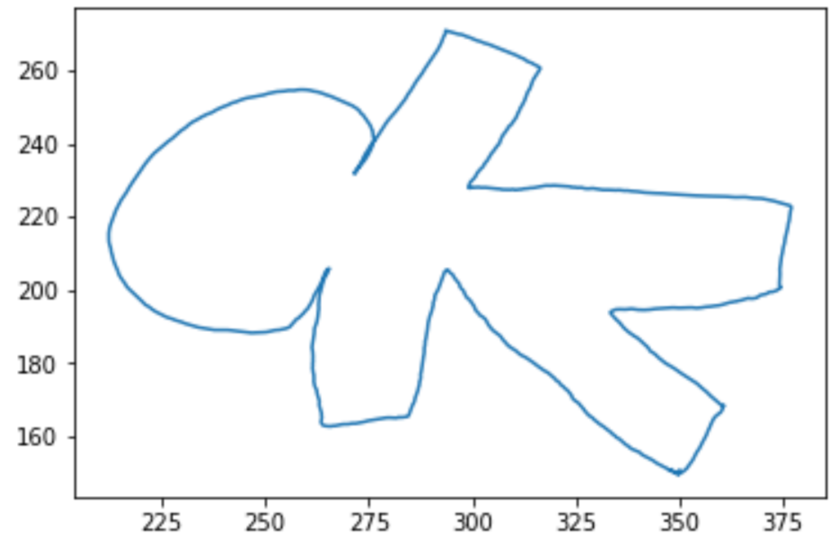
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

$a = 0.6; b = 1.1$



Matrices operating on data





Data set: *A*

Data set: *B*

Rotation

Notation and special matrices

- Square matrix: $m = n$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Zero matrix: $A_{ij} = 0$
- Identity matrix $[\boldsymbol{I}] = [\delta_{ij}]$
- Symmetric matrix: $A_{ij} = A_{ji}$ $\quad [\boldsymbol{A}] = [\boldsymbol{A}]^T$
- Permutation matrix:
  - Permutation of the identity matrix
  - Permutes (swaps) rows

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ a \\ b \end{pmatrix}$$

- Diagonal matrix: $A_{ij} = 0, \ \forall i, j \mid i \neq j$
- Triangular matrix:

Lower triangular: $L_{ij} = \begin{cases} L_{ij}, i \geq j \\ 0, \quad i < j \end{cases}$

Upper triangular: $U_{ij} = \begin{cases} U_{ij}, i \leq j \\ 0, \quad i > j \end{cases}$

More about matrices

- Rank: the rank of a matrix $\boldsymbol{A}$ is the dimension of the vector space generated by its columns, which is equivalent to the number of linearly independent columns of the matrix.
- Suppose $\boldsymbol{A}$ has shape $m \times n$:
  - $rank(\boldsymbol{A}) \leq \min(m, n)$
  - Matrix $\boldsymbol{A}$ is **full rank**: $rank(\boldsymbol{A}) = \min(m, n)$. Otherwise, matrix $\boldsymbol{A}$ is **rank deficient**.
- Singular matrix: a square matrix $\boldsymbol{A}$ is invertible if there exists a square matrix **B** such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$. If the matrix is not invertible, it is called singular.

Vector and Matrix Norms

Norms

What's a norm?

- A generalization of 'absolute value' to vectors.
- $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}_0^+$, returns a 'magnitude' of the input vector
- In symbols: Often written $\|\mathbf{x}\|$.

Define norm.

A function $\|\mathbf{x}\| : \mathbb{R}^n \to \mathbb{R}_0^+$ is called a norm if and only if

1. $\|\mathbf{x}\| > 0 \Leftrightarrow \mathbf{x} \neq \mathbf{0}$.
2. $\|\gamma \mathbf{x}\| = |\gamma| \, \|\mathbf{x}\|$ for all scalars γ.
3. Obeys triangle inequality $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$

Demo “Vector Norms”

Example of Norms

What are some examples of norms?

The so-called *p-norms*:

$$\left\| \begin{pmatrix} x_1 \\ \\ x_n \end{pmatrix} \right\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} \quad (p \geqslant 1)$$

$p = 1, 2, \infty$ particularly important

Unit Ball:

Set of vectors $\boldsymbol{x}$ with norm $||\boldsymbol{x}|| = 1$

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

Apply a norm!

How? Attempt 1:

$\text{Magnitude of error} \neq \|\text{true value}\| - \|\text{approximate value}\|$ **WRONG!**

Attempt 2:

$$\text{Magnitude of error} = \|\text{true value} - \text{approximate value}\|$$

Absolute and Relative Errors

What are the absolute and relative errors in approximating the location of Siebel center $(40.114, -88.224)$ as $(40, -88)$ using the 2-norm?

Matrix Norms

What norms would we apply to matrices?

- Easy answer: '*Flatten*' matrix as vector, use vector norm. This corresponds to an entrywise matrix norm called the Frobenius norm,

$$\|A\|_F := \sqrt{\sum_{i,j} a_{ij}^2}.$$

Matrix Norms

However, interpreting matrices as linear functions, what we are really interested in is the maximum amplification of the norm of any vector multiplied by the matrix,

$$\|A\| := \max_{\|\boldsymbol{x}\|=1} \|A\boldsymbol{x}\| \,.$$

These are called induced matrix norms, as each is associated with a specific vector norm $\|\cdot\|$.

Matrix Norms

The following are equivalent:

$$\max_{\|x\|\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|x\|\neq 0} \left\| A \underbrace{\frac{\mathbf{x}}{\|\mathbf{x}\|}}_{y} \right\| \overset{\|\mathbf{y}\|=1}{=} \max_{\|\mathbf{y}\|=1} \|A\mathbf{y}\| = \|A\| \,.$$

Logically, for each vector norm, we get a different matrix norm, so that, e.g. for the vector 2-norm $\|\mathbf{x}\|_2$ we get a matrix 2-norm $\|A\|_2$, and for the vector ∞-norm $\|\mathbf{x}\|_\infty$ we get a matrix ∞-norm $\|A\|_\infty$.

Induced Matrix Norms

$$\|\boldsymbol{A}\|_1 = \max_j \sum_{i=1}^{n} |A_{ij}|$$

Maximum absolute column sum of the matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_\infty = \max_i \sum_{j=1}^{n} |A_{ij}|$$

Maximum absolute row sum of the matrix $\boldsymbol{A}$

$$\|\boldsymbol{A}\|_2 = \max_k \sigma_k$$

σ_k are the singular value of the matrix $\boldsymbol{A}$

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

But also some more properties that stem from our definition:

1. $\|A\boldsymbol{x}\| \leqslant \|A\| \, \|\boldsymbol{x}\|$
2. $\|AB\| \leqslant \|A\| \, \|B\|$ (easy consequence)

Both of these are called submultiplicativity of the matrix norm.

Examples

Determine the norm of the following matrices:

1)

$$\left\| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\|_\infty$$

2)

$$\left\| \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\|_1$$

Matrix Norm Approximation

Suppose you know that for a given matrix A three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ for the vector norm $||\cdot||$,

$||\mathbf{x}|| = 2$, $||\mathbf{y}|| = 1$, $||\mathbf{z}|| = 3$,

and for corresponding induced matrix norm,

$||A\mathbf{x}|| = 20$, $||A\mathbf{y}|| = 5$, $||A\mathbf{z}|| = 90$.

What is the largest lower bound for $||A||$ that you can derive from these values?

Induced Matrix Norm of a Diagonal Matrix

What is the 2-norm-based matrix norm of the diagonal matrix

$$A = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}?$$

Induced Matrix Norm of an Inverted Diagonal Matrix

What is the 2-norm-based matrix norm of the **inverse** of the diagonal matrix

$$A = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}?$$