Eigenvalues and Eigenvectors

Eigenvalue problem

Let $\boldsymbol{A}$ be an $n \times n$ matrix:

$\boldsymbol{x} \neq \boldsymbol{0}$ is an **eigenvector** of $\boldsymbol{A}$ if there exists a scalar λ such that

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$

eigenvectors

where λ is called an **eigenvalue**.

eigenvalues

eigepairs

If $\boldsymbol{x}$ is an eigenvector, then $\alpha\boldsymbol{x}$ is also an eigenvector. Therefore, we will usually seek for **normalized eigenvectors**, so that

$\underset{\sim}{x} = \alpha \underset{\sim}{u}$

$A(\alpha u) = \lambda(\alpha u)$

$$\boxed{\|x\|_p = 1}$$

$\| x \|_2 = 1$

Note: When using Python, numpy.linalg.eig will normalize using $p=2$ norm.

How do we find eigenvalues?

Linear algebra approach:

$$\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$$
$$(\boldsymbol{A} - \lambda\,\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

Therefore the matrix $(\boldsymbol{A} - \lambda\,\boldsymbol{I})$ is singular $\Longrightarrow$ $\boxed{det(\boldsymbol{A} - \lambda\,\boldsymbol{I}) = 0}$

$p(\lambda) = det(\boldsymbol{A} - \lambda\,\boldsymbol{I})$ is the characteristic polynomial of degree n.

In most cases, there is no analytical formula for the eigenvalues of a matrix (Abel proved in 1824 that there can be no formula for the roots of a polynomial of degree 5 or higher) $\Longrightarrow$ **Approximate the eigenvalues numerically**!

Example

columns of A are L.D. $\Rightarrow$ rank(A) = 1

A is singular matrix

$\det(A - \lambda I) = 0$

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\det\begin{pmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{pmatrix} = 0$$

$p(\lambda) = (2-\lambda)^2 - 4 = 0 \longrightarrow 4 - 2(2)\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 4\lambda = 0$

$\lambda(\lambda - 4) = 0$ — $\lambda = 0$, $\lambda = 4$ } two distinct eigenvalues

eigenvectors

$(A - \lambda I)x = 0$

$$\begin{pmatrix} 2-0 & 1 \\ 4 & 2-0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$2u_1 + u_2 = 0$

$4u_1 + 2u_2 = 0$ } $u_2 = -2u_1$

$x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$\boxed{\lambda = 0}$

$$\begin{pmatrix} 2-4 & 1 \\ 4 & 2-4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-2u_1 + u_2 = 0$

$4u_1 - 2u_2 = 0$ } $u_2 = 2u_1$

$\underset{\sim}{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{\lambda = 4}$

2 L.I. eigenvectors

Diagonalizable Matrices

A $n \times n$ matrix $\boldsymbol{A}$ with n linearly independent eigenvectors $\boldsymbol{u}$ is said to be **diagonalizable**.

$$\boldsymbol{A}\,\boldsymbol{u}_1 = \lambda_1\,\boldsymbol{u}_1$$
$$\boldsymbol{A}\,\boldsymbol{u}_2 = \lambda_2\,\boldsymbol{u}_2,$$
$$\ldots$$
$$\boldsymbol{A}\,\boldsymbol{u}_n = \lambda_n\,\boldsymbol{u}_n,$$

$\lambda_1, u_1 \rightarrow Au_1 = \lambda_1 u_1$

$$\underset{\sim}{A}\underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{\underset{\sim}{U}} = \begin{bmatrix} | & | & & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \\ | & | & & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{\underset{\sim}{U}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\underset{\sim}{D}} \Rightarrow \underset{\sim}{A}\,\underset{\sim}{U} = \underset{\sim}{U}\,\underset{\sim}{D}$$

$\Downarrow$

full rank $\Rightarrow$ $\boxed{\underset{\sim}{A} = \underset{\sim}{U}\,\underset{\sim}{D}\,\underset{\sim}{U}^{-1}}$

diagonalizable!

Example

$$\boxed{\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}} \qquad det\begin{pmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{pmatrix} = 0$$

Solution of characteristic polynomial gives: $\underline{\lambda_1 = 4, \lambda_2 = 0}$

To get the eigenvectors, we solve: $\boldsymbol{A}\,\boldsymbol{x} = \lambda\,\boldsymbol{x}$

$$\begin{pmatrix} 2-(4) & 1 \\ 4 & 2-(4) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightarrow x = \begin{pmatrix} 0.447 \\ 0.894 \end{pmatrix}$$

$$\begin{pmatrix} 2-(0) & 1 \\ 4 & 2-(0) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rightarrow x = \begin{pmatrix} -0.447 \\ 0.894 \end{pmatrix}$$

normalized eigenvectors

⇓

linearly ind.

⇓

A is diag.

$$\boxed{\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{D}}\,\underline{\underline{U}}^{-1}}$$

$$\underline{\underline{U}} = \begin{bmatrix} 0.447 & -0.447 \\ 0.894 & 0.894 \end{bmatrix}$$

$$\underline{\underline{D}} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

Example

$\det(A) = 27 - (-36) = 63 \neq 0$

$\rightarrow$ NOT SINGULAR

The eigenvalues of the matrix:

$$\boldsymbol{A} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$$

$(A - \lambda I)x = 0$

$$\begin{pmatrix} 3-(-3) & -18 \\ 2 & -9-(-3) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are $\lambda_1 = \lambda_2 = -3.$

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Select the **incorrect** statement:

$$\left.\begin{aligned} 6u_1 - 18u_2 &= 0 \\ 2u_1 - 6u_2 &= 0 \end{aligned}\right\} \rightarrow \text{one eigenvector}$$

A) Matrix $\boldsymbol{A}$ is diagonalizable $\rightarrow$ False

B) The matrix $\boldsymbol{A}$ has only one eigenvalue with multiplicity 2 $\rightarrow$ True

C) Matrix $\boldsymbol{A}$ has only one linearly independent eigenvector $\rightarrow$ True

D) Matrix $\boldsymbol{A}$ is not singular $\rightarrow$ True

$\cancel{A = UDU^{-1}}$

only one eigenvector

Let's look back at diagonalization…

1) If a $n \times n$ matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{x}$ then $\boldsymbol{A}$ is diagonalizable, i.e.,

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$$

where the columns of $\boldsymbol{U}$ are the linearly independent normalized eigenvectors $\boldsymbol{x}$ of $\boldsymbol{A}$ (which guarantees that $\boldsymbol{U}^{-1}$ exists) and $\boldsymbol{D}$ is a diagonal matrix with the eigenvalues of $\boldsymbol{A}$.

2) If a $n \times n$ matrix $\boldsymbol{A}$ has less then n linearly independent eigenvectors, the matrix is called defective (and therefore not diagonalizable).

3) If a $n \times n$ **symmetric** matrix $\boldsymbol{A}$ has n distinct eigenvalues then $\boldsymbol{A}$ is diagonalizable.

A $n \times n$ symmetric matrix A with n distinct eigenvalues is diagonalizable.

$\lambda, u \rightarrow Au = \lambda u$

$\mu, v \rightarrow \boxed{Av = \mu v}$

A is diag

Suppose $\lambda, \boldsymbol{u}$ and $\mu, \boldsymbol{v}$ are eigenpairs of $\boldsymbol{A}$

$\lambda\, \boldsymbol{u} = \boldsymbol{A}\boldsymbol{u}$

$\mu\, \boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}$

$Au = \lambda u \longrightarrow$ vector

$v \cdot Au = \lambda\, v \cdot u \longrightarrow$ scalars

$A^T v \cdot u = \lambda\, v \cdot u$

A symmetric $\Rightarrow Av \cdot u = \lambda\, v \cdot u$

$A^T = A$

$\mu\, v \cdot u = \lambda\, v \cdot u$

$\underbrace{(\mu - \lambda)}_{\neq 0} (v \cdot u) = 0 \Rightarrow \boxed{v \cdot u = 0}$

orthogonal vectors $\Rightarrow$ L.I

Some things to remember about eigenvalues:

- Eigenvalues can have zero value
- Eigenvalues can be negative
- Eigenvalues can be real or complex numbers
- A $n \times n$ real matrix can have complex eigenvalues
- The eigenvalues of a $n \times n$ matrix are not necessarily unique. In fact, we can define the multiplicity of an eigenvalue.
- If a $n \times n$ matrix has n linearly independent eigenvectors, then the matrix is diagonalizable

How can we get eigenvalues numerically?

A, $n \times n$ $\longrightarrow$ $u_1, u_2, \ldots, u_n \Rightarrow$ L.I.

Assume that $\boldsymbol{A}$ is diagonalizable (i.e., it has n linearly independent eigenvectors $\boldsymbol{u}$). We can propose a vector $\boldsymbol{x}$ which is a linear combination of these eigenvectors:

$$\boxed{A u_i = \lambda_i u_i}$$

$$\boldsymbol{x} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n$$

$$\boxed{A x} = A\alpha_1 u_1 + A\alpha_2 u_2 + \cdots + A\alpha_n u_n$$

$$= \alpha_1 \lambda_1 u_1 + \alpha_2 \lambda_2 u_2 + \cdots + \alpha_n \lambda_n u_n$$

Assume:

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \ldots > |\lambda_n|$$

Power Iteration

$x_0 =$ ______
$x_1 =$ ______
$x_2 =$ ______
converge

Our goal is to find an eigenvector $\boxed{\boldsymbol{u}_i \text{ of } \boldsymbol{A}.}$ We will use an iterative process, where we start with an initial vector, where here we assume that it can be written as a linear combination of the eigenvectors of $\boxed{\boldsymbol{A}}$. $\longrightarrow$ diagonalizable

u_i all L.I.

$$\boldsymbol{x}_0 = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n$$

$$A x_0 = \alpha_1 \lambda_1 u_1 + \alpha_2 \lambda_2 u_2 + \cdots + \alpha_n \lambda_n u_n = x_1$$ after 1st iteration

$$A x_1 = A \alpha_1 \lambda_1 u_1 + A \alpha_2 \lambda_2 u_2 + \cdots + A \alpha_n \lambda_n u_n$$
$$= \alpha_1 \lambda_1 (\lambda_1 u_1) + \alpha_2 \lambda_2 (\lambda_2 u_2) + \cdots + \alpha_n \lambda_n (\lambda_n u_n)$$
$$= \alpha_1 \lambda_1^2 u_1 + \alpha_2 \lambda_2^2 u_2 + \cdots + \alpha_n \lambda_n^2 u_n = x_2$$ after 2nd iter.

$$A x_2 = \alpha_1 \lambda_1^3 u_1 + \alpha_2 \lambda_2^3 u_2 + \cdots + \alpha_n \lambda_n^3 u_n = x_3$$

$\vdots$

$$A x_{k-1} = \alpha_1 \lambda_1^k u_1 + \alpha_2 \lambda_2^k u_2 + \cdots + \alpha_n \lambda_n^k u_n = x_k$$

Power Iteration

$x_0 = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$A_{n \times n}$ $\underline{n}$

$$\boldsymbol{x}_k = (\lambda_1)^k \left[\alpha_1 \boldsymbol{u}_1 + \underbrace{\cancel{\alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k \boldsymbol{u}_2 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k \boldsymbol{u}_n}} \right]$$

Assume that $\alpha_1 \neq 0$, the term $\alpha_1 \boldsymbol{u}_1$ dominates the others when k is very large.

dominant

$|\lambda_1| > |\lambda_2| > |\lambda_3| \cdots |\lambda_n|$

Since $|\lambda_1| > |\lambda_2|$, we have $\left(\frac{\lambda_2}{\lambda_1}\right)^k \ll 1$ when k is large

Hence, as k increases, $\boldsymbol{x}_k$ converges to a multiple of the first eigenvector $\boldsymbol{u}_1$, i.e.,

$\lambda_1, \underset{\sim}{u_1}$

$k \to \infty \implies \underset{\sim}{x_k} \to \underline{\lambda_1^{k}} \alpha_1 \underset{\sim}{u_1}$

$\hookrightarrow$ large! $\|x_k\| \to$ grow