Sources of Error

Main source of errors in numerical computation:

- **Rounding error:** occurs when digits in a decimal point ($1/3 = 0.3333\ldots$) are lost (0.3333) due to a limit on the memory available for storing one numerical value.

- **Truncation error:** occurs when discrete values are used to approximate a mathematical expression (eg. the approximation $\sin(\theta) \approx \theta$ for small angles θ)

Truncation errors: using Taylor series to approximate functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + 2\,a_2\,x + 3\,a_3\,x^2 + 4\,a_4\,x^3 + \cdots$$

$$f''(x) = 2\,a_2 + (3\times2)a_3\,x + (4\times3)a_4\,x^2 + \cdots$$

$$f'''(x) = (3\times2)a_3 + (4\times3\times2)a_4\,x + \cdots$$

$$f'^{v}(x) = (4\times3\times2)a_4 + \cdots$$

$f(0) = a_0$

$f'(0) = a_1$

$f''(0) = \underbrace{2}a_2$

$f'''(0) = \underbrace{3\times2}a_3$

$f^{(iv)}(0) = \underbrace{4\times3\times2}\,a_4$

$\cdots$

$f^{(i)}(0) = i!\,a_i$

$$\boxed{a_i = \frac{f^{(i)}(0)}{i!}}$$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} a_i\, x^i \rightarrow \boxed{f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}\, x^i}$$

→ approximate function values

→ approximate derivatives

→ estimate erros of these approximations

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(x_o)}{3!}(x - x_o)^3 + \cdots$$

$$\boxed{f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i}$$

Use Taylor to approximate functions at given point

$x_0 = 1 \quad x = 4$

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4; f^{(n)}(1) = 0 \; \forall \, n \geq 3$$

Evaluate $f(4)$

$$f(4) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 \,//$$

$$f(4) = f(1) + f'(1)(4-1) + \frac{f''(1)}{2}(4-1)^2$$

$$= 2 + (-3)(3) + \frac{4}{2}(3)^2 \Rightarrow f(4) = 11 \,//$$

Example:

$f'(x)$

$f''(x)$

Given the function

$$f(x) = \frac{1}{(20x - 10)}$$

$f(0) = -\frac{1}{10}$

Write the Taylor approximation of degree 2 about point $x_o = 0$

$$f'(x) = \frac{-1(20)}{(20x-10)^2} \Rightarrow f'(0) = -\frac{1}{5}$$

$$f''(x) = \frac{+20(2)(20x-10)(20)}{(20x-10)^4} \Rightarrow f''(0) = -\frac{4}{5}$$

$$t_2(x) = -\frac{1}{10} - \frac{1}{5}x - \frac{4}{5}\frac{1}{2}x^2$$

Taylor Series – what is the error?

We cannot sum infinite number of terms, and therefore we have to **truncate**.

$x = h + x_0$

How **big is the error** caused by truncation? Let's write $h = x - x_0$

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots$$

$$\underbrace{f(x_0+h)}_{f(x)} = \underbrace{\sum_{i=0}^{n} \frac{f^{(i)}(x_0)h^i}{i!}}_{t_n(x)} + \underbrace{\sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^{(i)}}_{\text{error}}$$

truncated

$$|\text{error}| = |f(x) - t_n(x)|$$

$$= \left| \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i \right|$$

$$= \left| \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!} h^{n+1}}_{\text{dominant}} + \frac{f^{(n+2)}(x_0)}{(n+2)!} h^{n+2} + \cdots \right|$$

$$h \to 0 \Rightarrow x \to x_0$$

$$|\text{error}| \leq |M h^{n+1}| \Rightarrow |\text{error}| = O(h^{n+1})$$

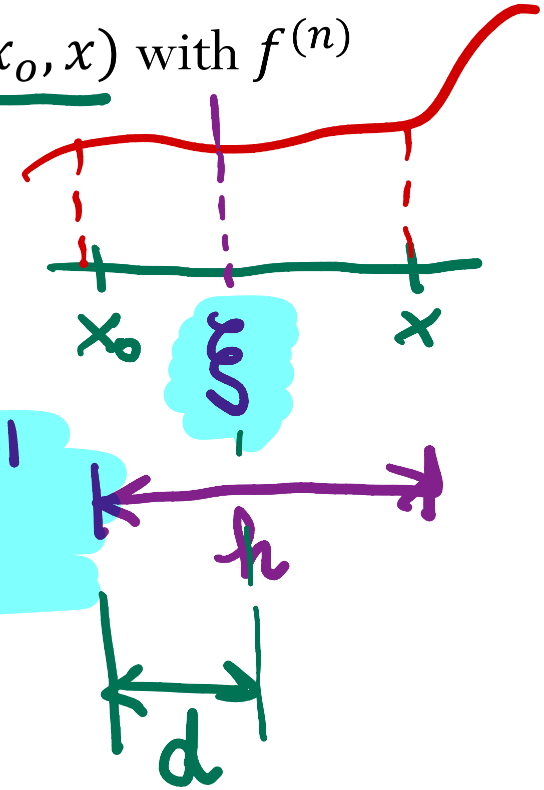
?

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

$$f(x) = t_n(x) + R(x)$$

$$\boxed{R(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!}(h)^i}$$



Then there exists a $\xi \in (x_o, x)$ so that

$$\boxed{R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(\xi - x_o)^{n+1}}$$

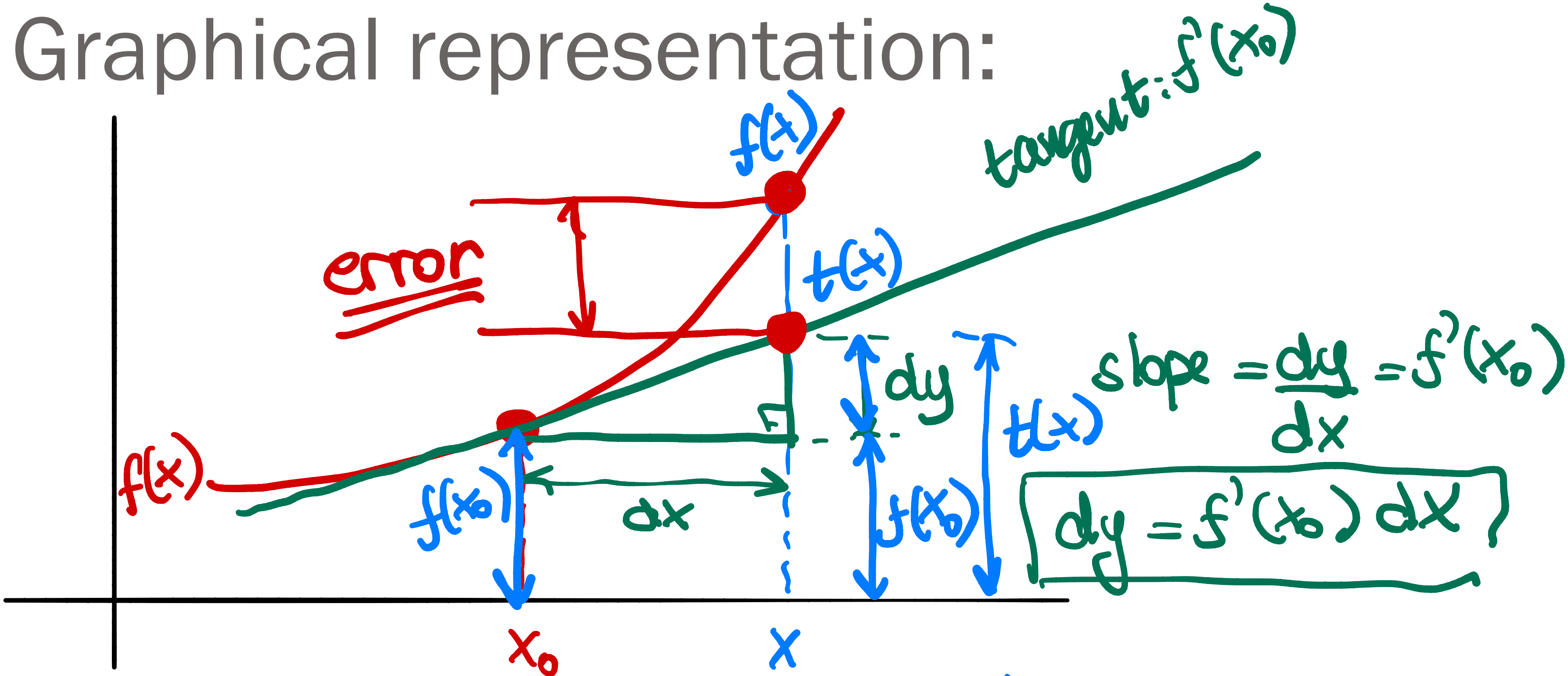
$$\boxed{|R| \leq \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}\right| h^{n+1}}$$

$$|R| \leq M h^{n+1}$$

$$d \leq h$$

$$|\xi - x_0| \leq h$$

Graphical representation:

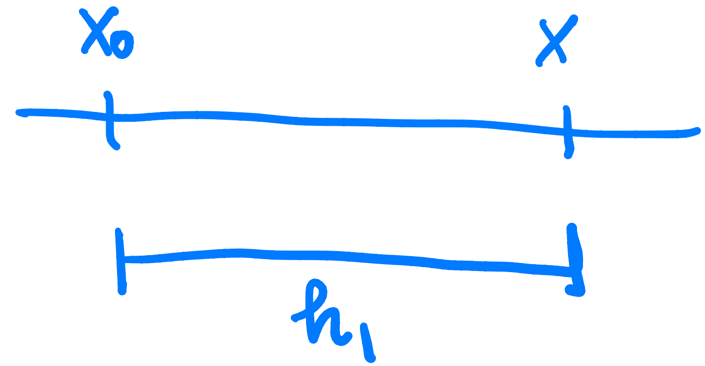


$$\underbrace{t(x)} = f(x_0) + dy = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{degree } 1}$$

$$e = O(h^{n+1}) \qquad \text{error} = O(h^2)$$

How can we use the known asymptotic behavior of the error?

$$f(x) \longrightarrow t_n(x)$$



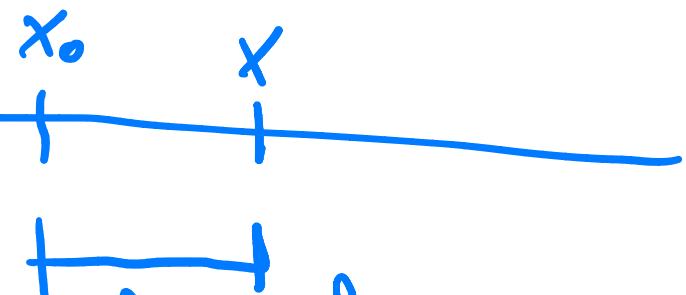
$$e_1 \Rightarrow h_1$$

$$e = O(h^{n+1})$$

$$e_1 \propto h_1^{n+1}$$

$$e_2 \propto h_2^{n+1}$$

$$\frac{e_1}{e_2} = \left(\frac{h_1}{h_2}\right)^{n+1}$$



$$h_2 = \frac{h_1}{2}$$

$$e_2 = \left(\frac{h_2}{h_1}\right)^{n+1} e_1$$

Making error predictions

Suppose you expand $\sqrt{x-10}$ in a Taylor polynomial of degree 3 about the center $x_0 = 12$. For $h_1 = 0.5$, you find that the Taylor truncation error is about 10^{-4}.

What is the Taylor truncation error for $h_2 = 0.25$?

$$f(x) = \sqrt{x-10} \longrightarrow t_3(x) \qquad x_0 = 12$$

$$h_1 = 0.5 \longrightarrow e_1 = 10^{-4}$$

$$h_2 = 0.25 \longrightarrow e_2 = ?$$

$$\frac{e_1}{e_2} = \left(\frac{h_1}{h_2}\right)^4 \longrightarrow e_2 = \left(\frac{h_2}{h_1}\right)^4 e_1$$

Using Taylor approximations to obtain derivatives

Let's say a function has the following Taylor series expansion about $x = 2$.

$$f(x) = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4 - \frac{5}{4}(x-2)^6 + \frac{25}{32}(x-2)^8 + O\left((x-2)^9\right)$$

$$\boxed{t_4 = \frac{5}{2} - \frac{5}{2}(x-2)^2 + \frac{15}{8}(x-2)^4}$$

$$t_4' = -5(x-2) + \frac{15}{2}(x-2)^3$$

$$t'(2.3) = -1.2975$$

$$t'(3.0) =$$