# Video 1: Error Definition

# Errors in Numerical Methods

- Every result we compute in Numerical Methods contain errors!

- We always have them… so our job? Reduce the impact of the errors

Main source of errors in numerical computation:

- **Rounding error:** occurs when digits in a decimal point ($1/3 = 0.3333...$) are lost (0.3333) due to a limit on the memory available for storing one numerical value.

- **Truncation error:** occurs when discrete values are used to approximate a mathematical expression (eg. the approximation $\sin(\theta) \approx \theta$ for small angles $\theta$)

# Evaluating the Error

$x$ : true value

$\hat{x}$ : approx.

- How can we model the error?

$$\hat{x} = x + \Delta x$$

- **Absolute error:** $e_a = |\hat{x} - x|$

- **Relative error:** $e_r = \dfrac{|\hat{x} - x|}{|x|}$

- Absolute errors can be misleading, depending on the magnitude of the true value $x$.

- For example, let's assume an absolute error $\Delta x = 0.1$

  - $x = 10^5 \rightarrow$ $10^5 + 0.1 =$
    (accurate result)

  - $x = 10^{-5} \rightarrow$ $10^{-5} + 10^{-1} =$
    (inaccurate result)

- Relative error is independent of magnitude.

You are tasked with measuring the height of a tree which is known to be exactly 170 ft tall. You later realized that your measurement tools are somewhat faulty, up to a relative error of 10%. What is the maximum measurement for the tree height?

$x = 170\,\text{ft}$

$\hat{x} = ?$

$e_r = 0.1$

$$e_r = \frac{|x - \hat{x}|}{|x|}$$

$$|x - \hat{x}| = e_r |x|$$

$\hat{x} = x(1 + e_r)$

$\hat{x} = x(1 - e_r)$

$$\boxed{\hat{x} = 170(1.1) = 187\,\text{ft}}$$

You are tasked with measuring the height of a tree and you get the measurement as 170 ft tall. You later realized that your measurement tools are somewhat faulty, up to a relative error of 10%. What is the minimum height of the tree?

$$\hat{x} = 170 \text{ ft}$$

$$e_r = 0.1$$

$$x = ?$$

$$e_r = \frac{|x - \hat{x}|}{|x|}$$

$$x = \frac{\hat{x}}{1 - e_r} \Rightarrow \text{max} \Rightarrow x = \frac{170}{0.9} = 188.8 \text{ ft}$$

$$x = \frac{\hat{x}}{1 + e_r} \Rightarrow \text{min} \Rightarrow x = \frac{170}{1.1} = 154.5 \text{ ft}$$

# Video 2: Significant Figures

# Significant digits

**Significant figures** of a number are digits that carry meaningful information. They are digits beginning to the leftmost nonzero digit and ending with the rightmost “correct” digit, including final zeros that are exact.

The number 3.14159 has ___6___ significant digits.

The number ~~0.000~~35 has ___2___ significant digits.

The number 0.000350 has ___3___ significant digits.

Suppose $x$ is the true value and $\tilde{x}$ the approximation.

**The number of significant figures tells us about how many positions of $x$ and $\hat{x}$ agree.**

**Suppose the true value is**

$$x = 3.141592653$$

**And the approximation is**

$$\hat{x} = 3.14$$

We say that $\hat{x}$ has 3 significant figures of $x$

Let's try the same for:

2) $\hat{x} = 3.14159$ → We say that $\hat{x}$ has 6 significant figures of $x$

3) $\hat{x} = 3.1415$ → We say that $\hat{x}$ has 4 significant figures of $x$

**What happened here?**

$\hat{x}$ has $\boldsymbol{n}$ **significant figures** of $x$ if $|x - \hat{x}|$ has zeros in the first $n$ decimal places counting from the leftmost nonzero (leading) digit of $x$, followed by a digit from 0 to 4.

$$\boxed{|x - \hat{x}| \leq 5 \times 10^{-n}}$$

$$x = 3.141592653$$

$$\underline{\hat{x} = 3.14159} \longrightarrow |x - \hat{x}| = \underbrace{0.000002653}_{6\ \text{zeros}} \rightarrow \hat{x} \text{ has } 6 \text{ sigfigs of } x$$

$$= 2.653 \times 10^{-6}$$

$$\underline{\hat{x} = 3.1415} \longrightarrow |x - \hat{x}| = \underbrace{0.000092653}_{4\ \text{zeros}} \rightarrow \hat{x} \text{ has } 4 \text{ sigfigs of } x$$

$$= 0.92653 \times 10^{-4}$$

$$\underline{\hat{x} = 3.1416} \longrightarrow |x - \hat{x}| = \underbrace{0.000007347}_{5\ \text{zeros}} \rightarrow \hat{x} \text{ has } 5 \text{ sigfigs of } x$$

$$= 0.7347 \times 10^{-5}$$

So far, we can observe that $|x - \hat{x}| \leq 5\times10^{-n}$.

Note that the exact number in this example can be written in the scientific notation form $x = q\times10^{0}$.

$3.1415\ldots \times 10^{0}$

**What happens when the exponent is not zero?**

We use the relative error definition instead!

$x = q \times 10^{p}$ $\quad \hat{x} = \hat{q} \times 10^{p}$

$|x - \hat{x}| = |q - \hat{q}| \times 10^{p}$

$e_r = \dfrac{|x - \hat{x}|}{|x|} = \dfrac{|q - \hat{q}| \times \cancel{10^{p}}}{|q| \times \cancel{10^{p}}} \leq \dfrac{5\times10^{-n}}{|q|} \leq 5\times10^{-n}$

$1 \leq |q| < 10$

$$\boxed{e_r \leq 5\times10^{-n}}$$

**Accurate to $n$ significant** digits means that you can trust a total of $n$ digits. Accurate digits is a measure of relative error.

$n$ is the number of accurate significant digits

Relative error: $\boxed{error = \frac{|x_{exact} - x_{approx}|}{|x_{exact}|} \le 5\times10^{-n}}$ $\le 10^{-n+1}$

**In general, we will use the rule-of-thumb:**

$$\boxed{relative\ error \le 10^{-n+1}}$$

**Rule-of-thumb:**

$$\boldsymbol{relative\ error \leq 10^{-n+1}}$$

A) For example, if relative error is $10^{-2}$ then $\hat{x}$ has at most 3 significant figures of $x$

$10^{-2}$ $\quad 10^{-n+1}$ $\quad -n+1=-2$

$n=3$

B) After rounding, the resulting number has 5 accurate digits. What is the tightest estimate of the upper bound on my relative error?

$n=5 \quad e_r \leq 10^{-n+1} = 10^{-5+1} \Rightarrow e_r \leq 10^{-4}$

# Video 3: Understanding Plots

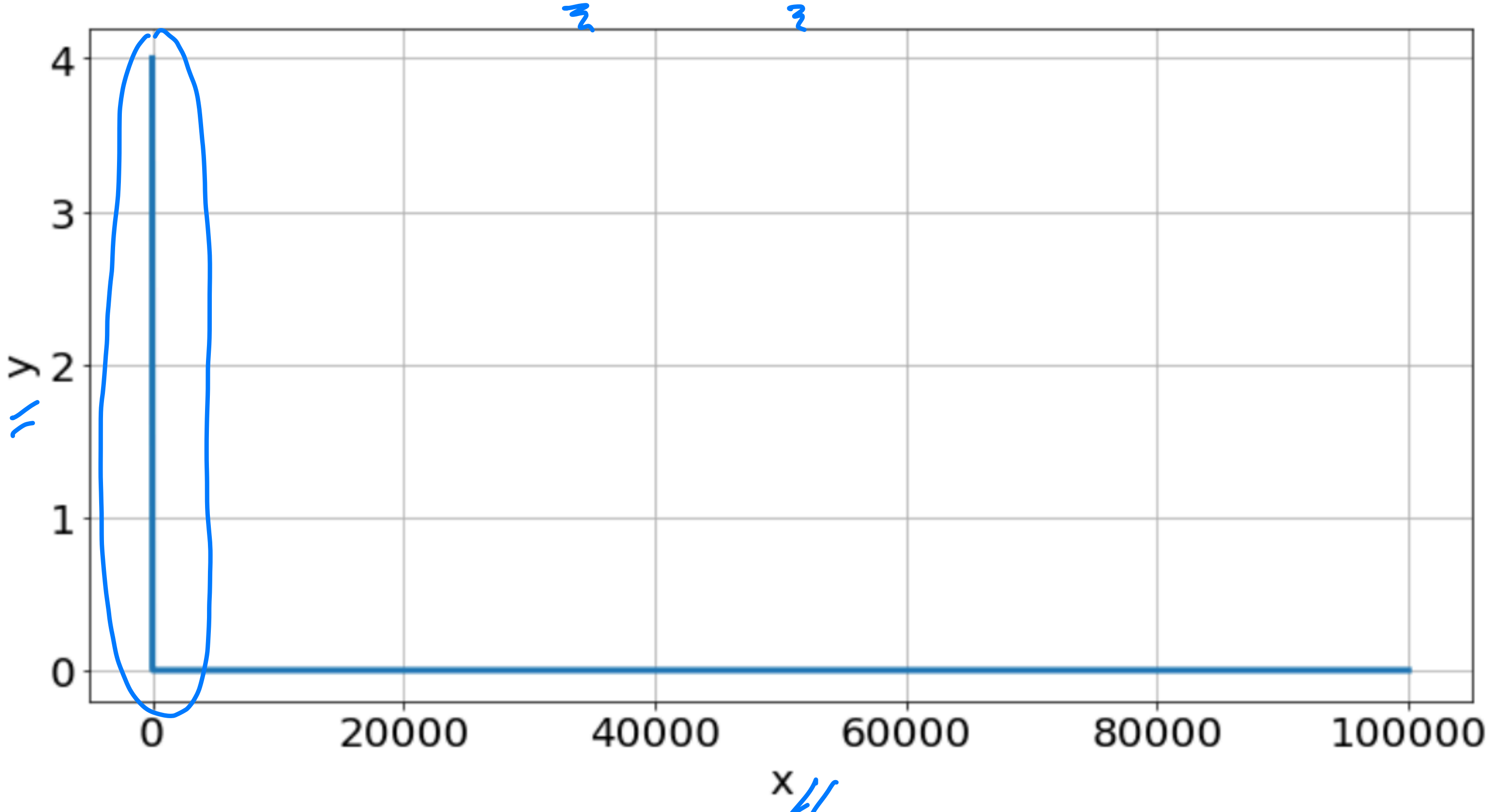
- Power functions: $y = a\,x^b$

$$\underbrace{\log(y)} = \log(ax^b) = \log(a) + \log(x^b)$$

$$= \underbrace{\log(a)} + b\,\underbrace{\log(x)}$$

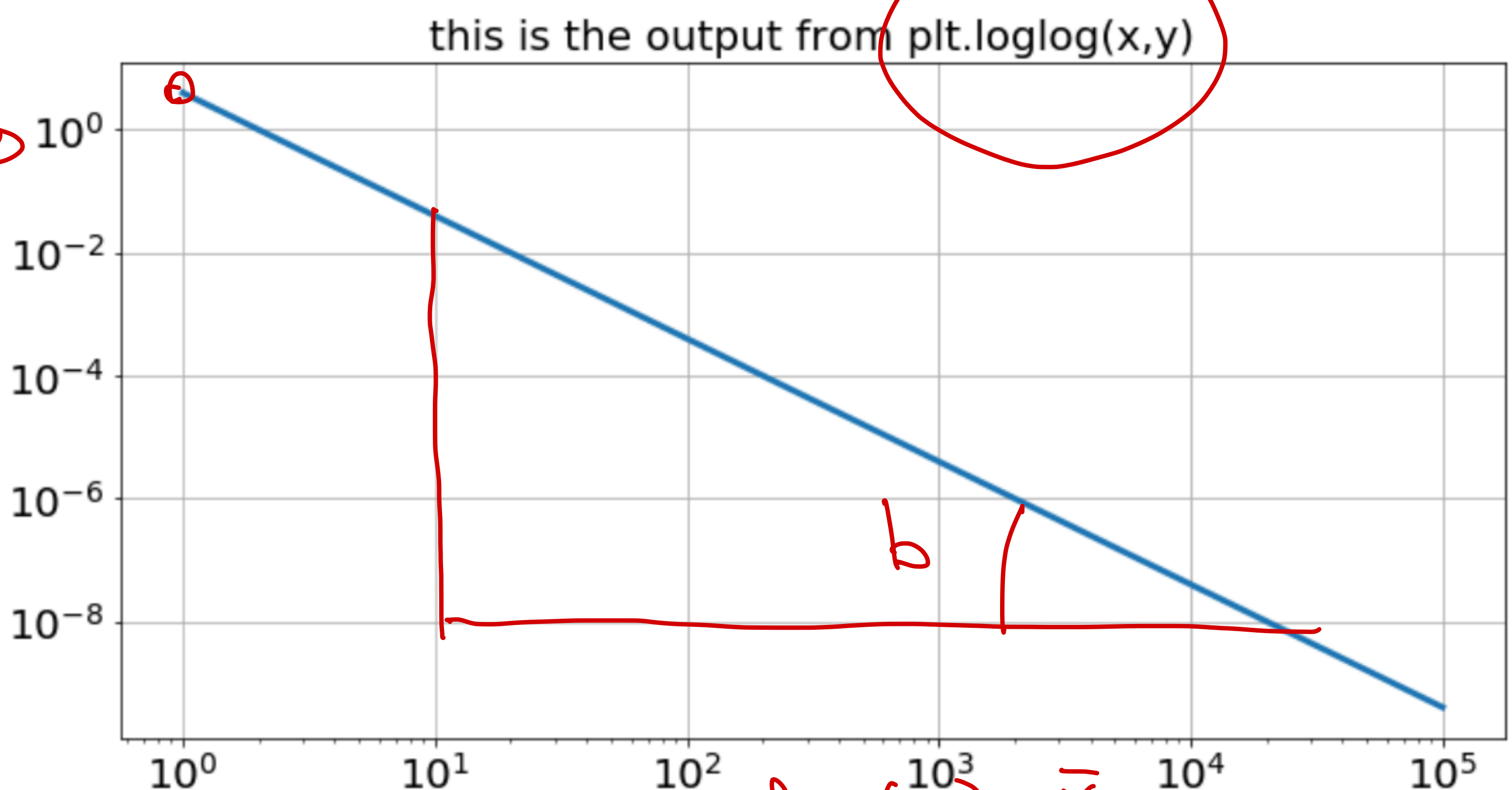
$$\boxed{\bar{y} = \bar{a} + b\,\bar{x}}$$

$a = 4$ and b $= -2$



- Power functions: $y = a\,x^b$

$$\bar{y} = \bar{a} + b\,\bar{x}$$




$\log(y) = \bar{y}$

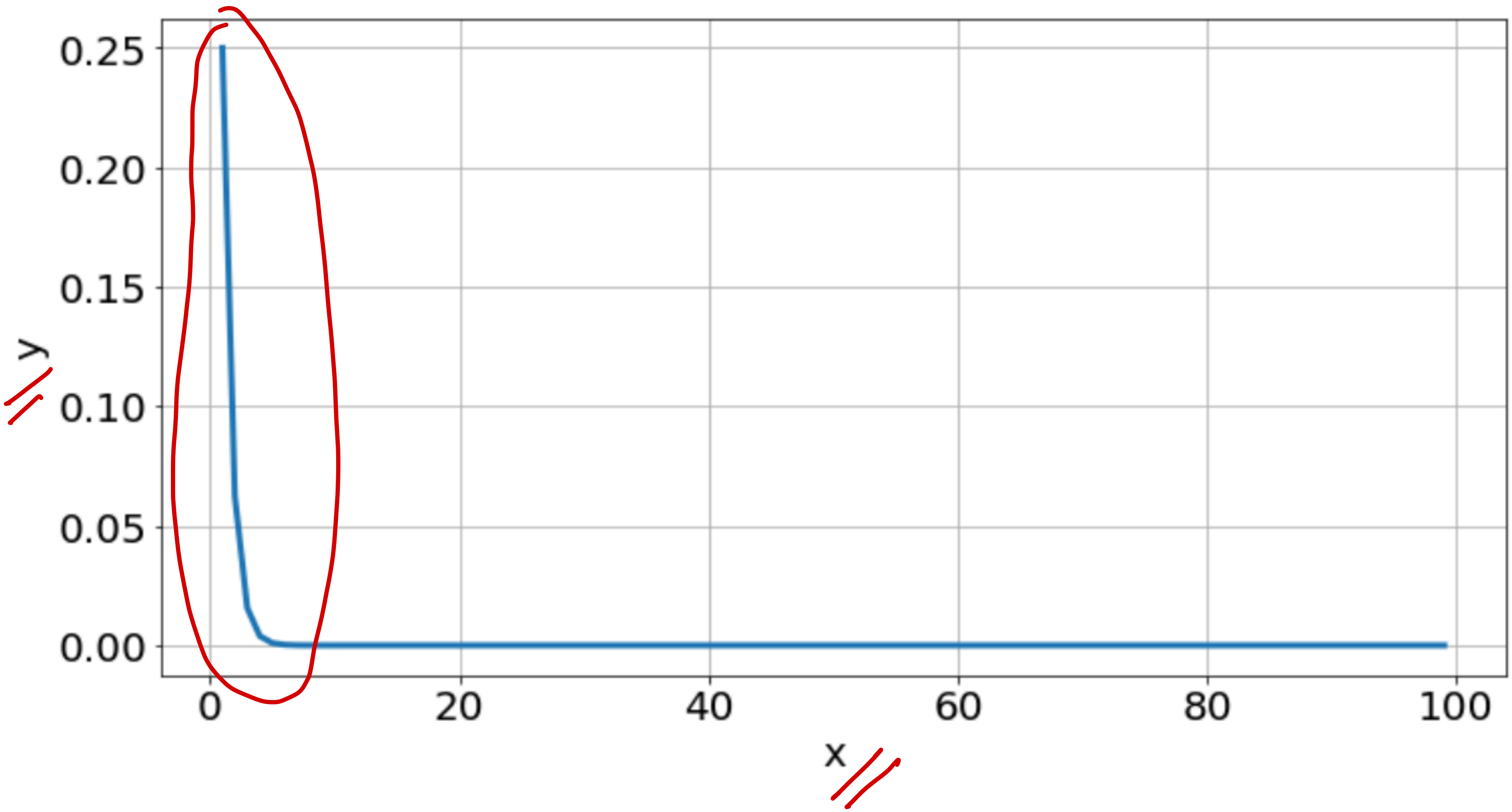
$\log(x) = \bar{x}$

- Exponential functions: $y = a\,b^x$

$$\underbrace{\log(y)}_{\bar{y}} = \log(ab^x) = \log(a) + \log(b^x)$$

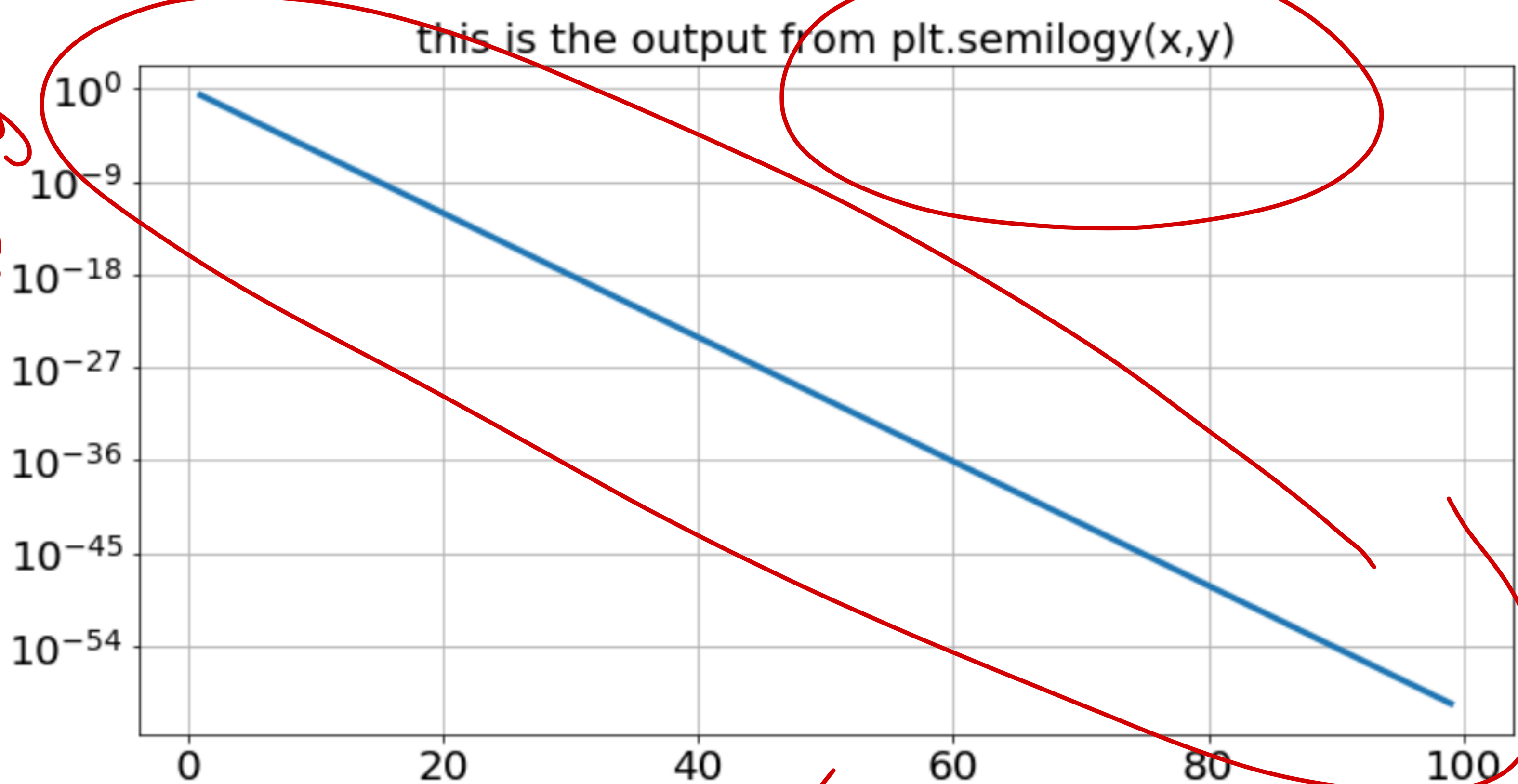
$$= \underbrace{\log(a)}_{\bar{a}} + x\,\underbrace{\log(b)}_{\bar{b}}$$

$a = 4$ and $b = -1$



- Exponential functions: $y = a\, b^{x}$

$$\bar{y} = \bar{a} + \bar{b}\, x$$




$\log(y) = \bar{y}$

$x$

# Video 4: Big-O notation

# Complexity: Matrix-matrix multiplication

For a matrix with dimensions $n \times n$, the computational complexity can be represented by a power function:

$n = 10 \longrightarrow t_1 = ?$

$n = 20 \longrightarrow t_2 = ?$

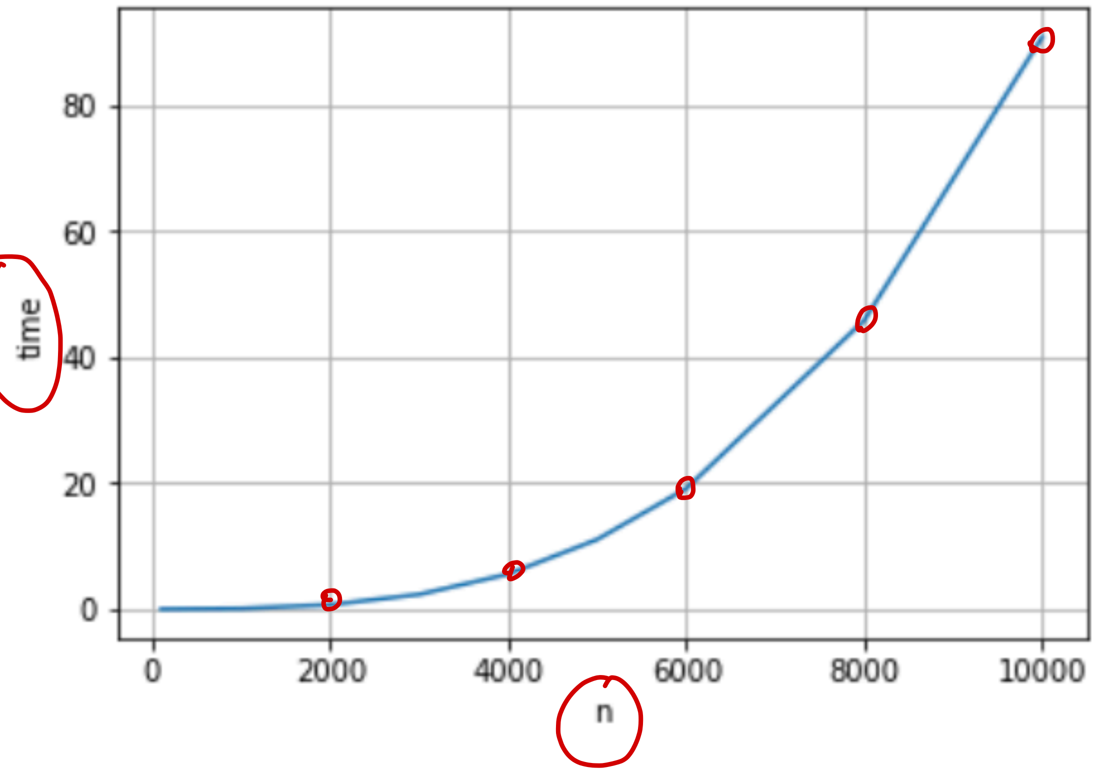
$\vdots \qquad \vdots$

$$time = c\, n^{a}$$

We could count the total number of operations to determine the value of the constants above, but instead, we will get an estimate using a numerical experiment where we perform several matrix-matrix multiplications for vary matrix sizes, and store the time to take to perform the operation.

For a matrix with dimensions $n \times n$, the computational complexity can be represented by a power function:

$$time = c\, n^{a}$$



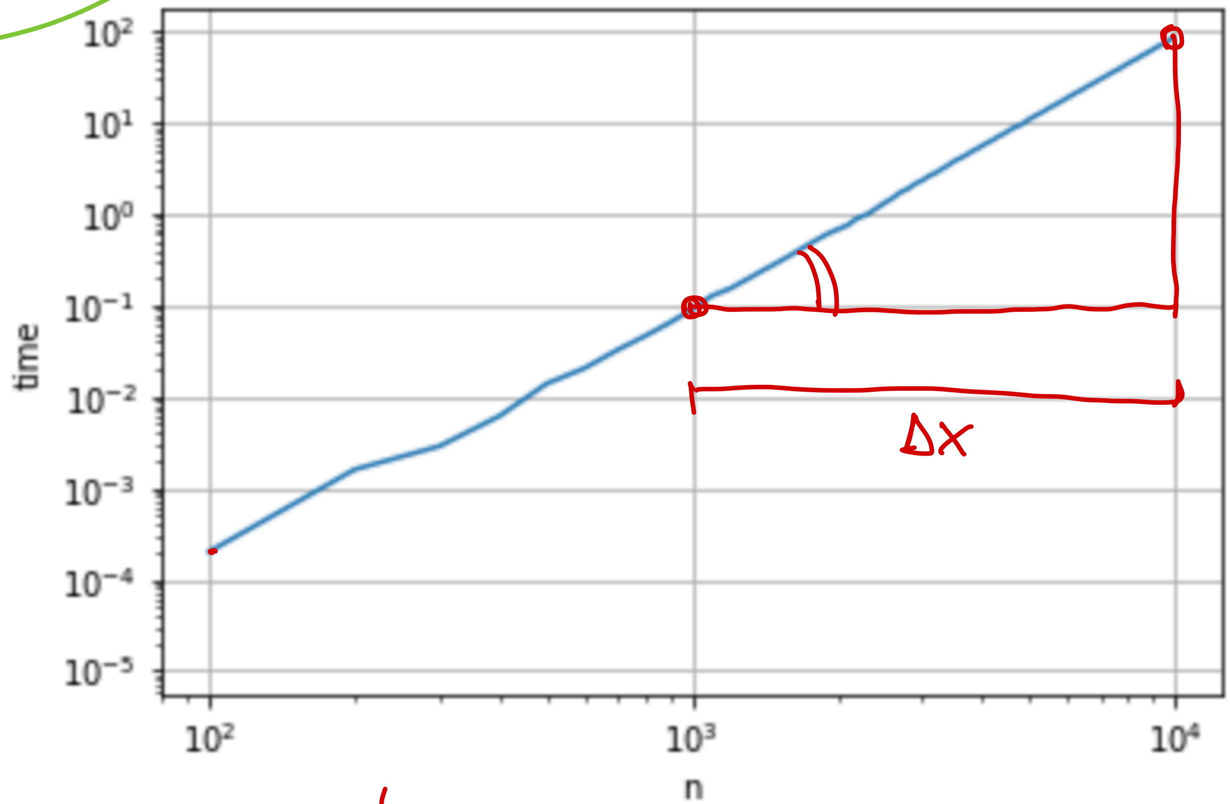

We can represent the power function above as a straight line using a log-log plot!

Power functions are represented by straight lines in a log-log plot, where the coefficient $a$ is determined by the slope of the line.

$$\boxed{time = c\, n^3}$$

$$time = c\ n^{\boxed{a}}$$

$$slope = \frac{\log(10^2) - \log(10^{-1})}{\log(10^4) - \log(10^3)} = \frac{(2\log(10) + 1\log(10))}{(4-3)\log(10)}$$



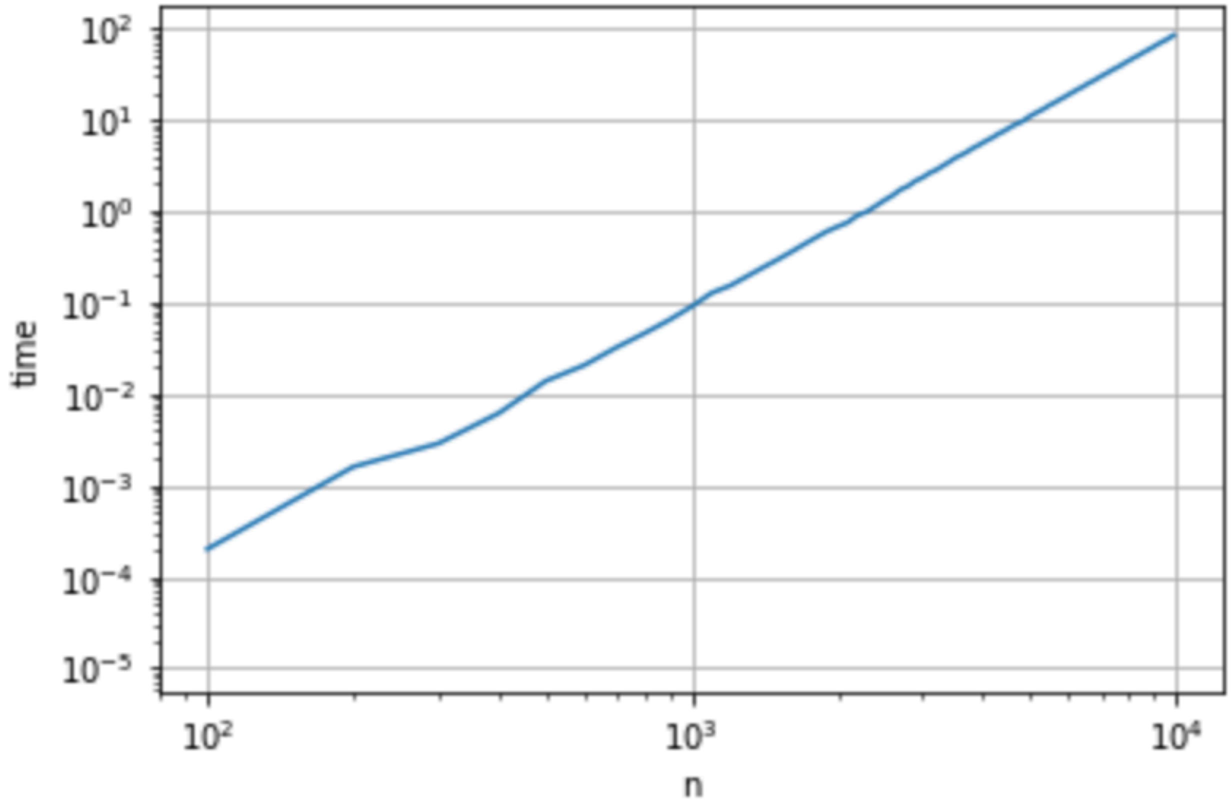

$$\Delta y \quad \boxed{= \frac{3}{1}}$$

$$slope = 3 \longrightarrow a = 3$$

Instead of predicting time using $time = c\, n^{a}$ , we can use the big-O notation to write

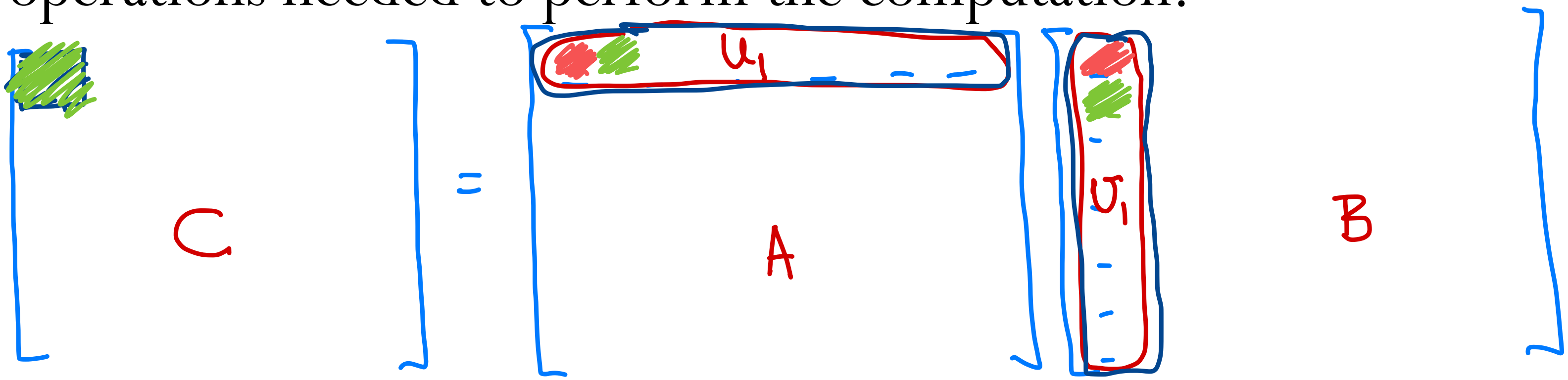
$$time = O(n^{a})$$

where $a$ can be obtained from the slope of the straight line.
For a matrix-matrix multiplication, what is the value of $a$?



$C = A \times B$

We can also get the complexity by counting the number of operations needed to perform the computation:




$n \times n$ $\quad$ $n \times n$ $\quad$ $n \times n$

$$C_{11} = u_1 \cdot v_1 = \begin{array}{l} n \text{ multiplications} \\ n \text{ summations} \\ \hline 2n \text{ operations} \end{array}$$

$$\left.\begin{array}{l} (u_1)_1 * (v_1)_1 \\ (u_1)_2 * (v_1)_2 \\ \vdots \\ (u_1)_n * (v_1)_n \end{array}\right\} +$$

$n^2$ entries in $C \longrightarrow n^2(2n)$ operations

$t \propto 2n^3$

# Big-Oh notation

Let $f$ and $g$ be two functions. Then

$$f(x) = O\big(g(x)\big) \text{ as } x \to \infty$$

If an only if there is a positive constant M such that for all sufficiently large values of $x$, the absolute value of $f(x)$ is at most multiplied by the absolute value of $g(x)$. In other words, there exists a value $M$ and some $x_0$ such that:

$$|f(x)| \leq \boxed{M\,|g(x)|} \quad \forall\, x \geq x_0$$

# Example:

dominant

Consider the function $f(x) = 2x^2 + 27x + 1000$

$x \to \infty$

$|f(x)| \le M(\underbrace{g(x)}) \quad x \to \infty$

$f(x) \le M$

$f(x) \le M(2x^2) \Longrightarrow f(x) = O(x^2)$ ✓

$f(x) = O(2x^2)$ ✓

# Accuracy: approximating sine function

The sine function can be expressed as an infinite series:

$$f(x) = \sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

(we will discuss these approximations later)

Suppose we approximate $f(x)$ as $\tilde{f}(x) = x$

We can define the error as:

$$E = |f(x) - \tilde{f}(x)| = \left| -\frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right|$$

$x \to 0$

Or we can use the Big-O notation to say:

$$\boldsymbol{E = O\left(x^3\right)}$$

$E \leq M\left(\left|\frac{x^3}{6}\right|\right)$

$E = O(x^3)$

# Big-Oh notation (continue)

Let $f$ and $g$ be two functions. Then

$$f(x) = O\big(g(x)\big) \text{ as } x \to a$$

If an only if there exists a value $M$ and some $\delta$ such that:

$$|f(x)| \le M\,|g(x)| \quad \forall x \;\; where\; 0 < |x - a| < \delta$$

# Same example…

Consider the function $f(x) = 2x^2 + 27x + 1000$

$$x \to 0$$

$f(x) \leq M(1000)$

$\boxed{f(x) = O(1)}$

# Iclicker question

Suppose that the truncation error of a numerical method is given by the following function:

$$E(h) = 5h^2 + 3h$$

Which of the following functions are Oh-estimates of $E(h)$ as $h \to 0$

1) ~~$O(5h^2)$~~
2) $\boxed{O(h)}$ ✓
3) $O(5h^2 + 3h)$ ✓
4) ~~$O(h^2)$~~

$\underbrace{5h^2+3h}$ $\leq M(5h^2)$ ✗ $h \to 0$

$5h^2+3h \leq M(h)$ $O(h)$ ✓

$5h^2+3h \leq M(5h^2+3h)$ ✓

$5h^2+3h \leq M(h^2)$ ✗

# Iclicker question

grow

Suppose that the complexity of a numerical method is given by the following function:

$$c(n) = 5n^2 + 3n$$

Which of the following functions are Oh-estimates of $c(n)$ as $n \to \infty$

1) $O(5n^2 + 3n)$
2) $O(n^2)$
3) $O(n^3)$
4) ~~$O(n)$~~

$(5n^2 + 3n) \leq M(5n^2 + 3n)$ ✓

$(5n^2 + 3n) \leq M(n^2)$ ✓

$(5n^2 + 3n) \leq M(n^3)$ ✓

$(5n^2 + 3n) \leq M(n)$ ✗

# Video 5: Making predictions

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} B \end{bmatrix}\begin{bmatrix} C \end{bmatrix}$$

$m = 10, 20$

$n = 10^5, 10^8$

Suppose the computational complexity of a numerical method is given by $O(n^3)$. When $n = 1000$, it was observed that the method takes 10 seconds to complete. You would like to run the same method for $n = 10000$. What is an estimate of the time for completion of the larger problem?

$$t = c\,n^3$$

$$n_1 = 1000 \longrightarrow t_1 = 10 \text{ seconds}$$

$$n_2 = 10^4 \longrightarrow t_2 = ?$$

$$t_1 = c\,n_1^3 \qquad t_2 = c\,n_2^3 \longrightarrow \frac{t_1}{t_2} = \frac{n_1^3}{n_2^3} \Rightarrow \boxed{t_2 = \left(\frac{n_2}{n_1}\right)^3 t_1}$$

$$t_2 = \left(\frac{10^4}{10^3}\right)^3 10 \text{ seconds}$$

$$\boxed{t_2 = 10^4 \text{ sec}}$$

Check the course notes: Error - BigO Role of Constants

# Video 6: Rates of convergence

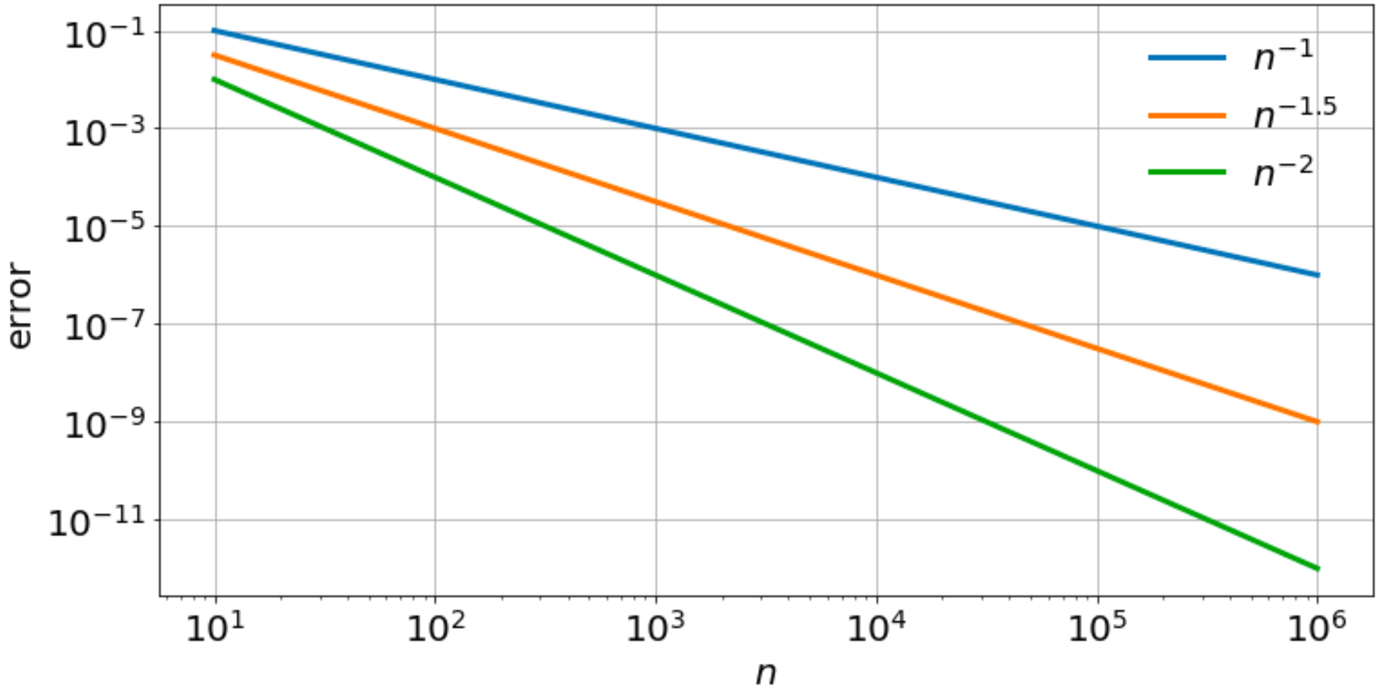
# Rates of convergence

**1) Algebraic convergence:** $error \sim \frac{1}{n^{\alpha}}$ or $O\left(\frac{1}{n^{\alpha}}\right)$

**Algebraic growth:** $time \sim n^{\alpha}$ or $O(n^{\alpha})$

$\alpha$: Algebraic index of convergence

A sequence that grows or converges algebraically is a straight line in a log-log plot.



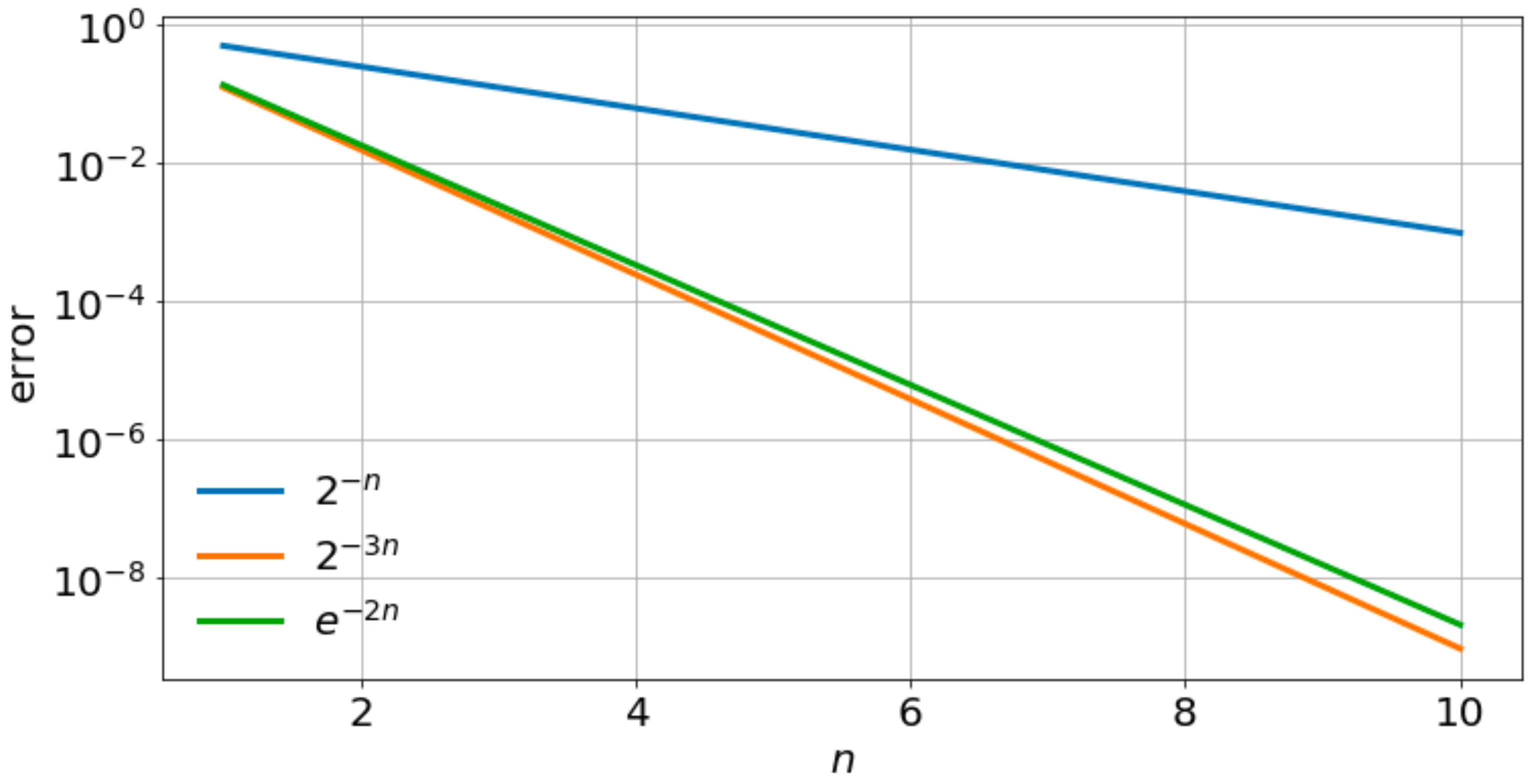

Demo "Exponential, Algebraic and Geometric convergence"

# Rates of convergence

**2) Exponential convergence:** $error \sim e^{-\alpha n}$ or $O(e^{-\alpha n})$

**Exponential growth:** $time \sim e^{\alpha n}$ or $O(e^{\alpha n})$

A sequence that grows or converges exponentially is a straight line in a linear-log plot.



# Rates of convergence

Exponential growth/convergence is much faster than algebraic growth/convergence.