Truncation errors: using Taylor series to approximate functions

Approximating functions using polynomials:

Let's say we want to approximate a function $f(x)$ with a polynomial

$$f(x) = a_o + a_1\, x + \; a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

For simplicity, assume we know the function value and its derivatives at $x_o = 0$ (we will later generalize this for any point). Hence,

$$f'(x) = a_1 + \; 2\, a_2\, x + 3\, a_3\, x^2 + 4\, a_4\, x^3 + \cdots$$

$$f''(x) = 2\, a_2 + \; (3{\times}2) a_3\, x + (4{\times}3) a_4\, x^2 + \cdots$$

or $f^{(i)} = i!\, a_i$

$\hookrightarrow \boxed{a_i = f^{(i)} / i!}$

$$f'''(x) = (3{\times}2) a_3 \; + (4{\times}3{\times}2) a_4\, x + \cdots$$

$f^{(i)} = \left(i \times (i-1) \times (i-2) \times \ldots \times 1 \right) a_i$

$$f'^{v}(x) = (4{\times}3{\times}2) a_4 + \cdots$$

$f(0) = a_o$

$f'(0) = a_1$

$f''(0) = 2\, a_2$

$f'''(0) = (3{\times}2)\, a_3$

$f'^{v}(0) = (4{\times}3{\times}2)\, a_4$

Taylor Series

Taylor Series approximation about point $x_o = 0$

$$f(x) = a_o + a_1\, x + a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} a_i\, x^i$$

$$\Rightarrow \boxed{f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!} x_i}$$

→ approximate function values

→ approximate derivatives

→ estimating errors

Taylor Series

In a more general form, the Taylor Series approximation about point x_o is given by:

$$f(x) = f(x_o) + f'(x_o)(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(0)}{3!}(x - x_o)^3 + \cdots$$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_o)}{i!}(x - x_o)^i$$

Example:

Assume a finite Taylor series approximation that converges everywhere for a given function $f(x)$ and you are given the following information:

$$f(1) = 2; f'(1) = -3; f''(1) = 4; \; f^{(n)}(1) = 0 \; \forall \, n \; \geq 3$$

Evaluate $f(4)$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cancel{\frac{f'''(x_0)}{3!}(x-x_0)^3} + \cdots$$

Make $x = 4$ and $x_0 = 1$

$$f(4) = f(1) + f'(1)(4-1) + \frac{f''(1)}{2}(4-1)^2 = 2 + (-3)(4-1) + \frac{4}{2}(4-1)^2$$

$$= 2 - 9 + 18 \Rightarrow \boxed{f(4) = 11}$$

Taylor Series

We cannot sum infinite number of terms, and therefore we have to **truncate**.

How **big is the error** caused by truncation? Let's write $\overbrace{h = x - x_o}^{x = h + x_0}$

$$f(x_0+h) = f(x_0) + f'(x_0)\,h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!}h^3 + \dots$$

$$\underbrace{f(x_0+h)}_{\text{exact}} = \underbrace{\sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i}_{\text{truncated part}} + \underbrace{\sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i}_{\text{what we are neglecting}}$$

$f(x)$

(Taylor approximation of degree n)

$t_n(x)$

error

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

error = exact - approximation

$$\text{error} = f(x) - t_n(x) = \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_0)}{i!} h^i$$

$$= \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!} h^{n+1}}_{\text{dominant term when } h \to 0 \text{ (or } x \to x_0\text{)}} + \frac{f^{(n+2)}(x_0)}{(n+2)!} h^{n+2} + \cdots$$

$$\text{error} \leq M h^{n+1} \quad \text{or} \quad \text{error} = O\left(h^{n+1}\right)$$

Taylor series with remainder

Let f be $(n+1)$-times differentiable on the interval (x_o, x) with $f^{(n)}$ continuous on $[x_o, x]$, and $h = x - x_o$

error = exact - approximation

Remainder Theorem :

$$R_n(x) = f(x) - t_n(x)$$

$$= \sum_{i=n+1}^{\infty} \frac{f^{(i)}(x_o)}{i!} h^i$$

$$\boxed{R_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(\xi - x_0)^{n+1}}$$

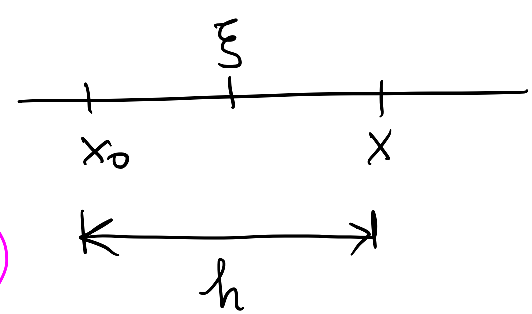
where $\xi \in (x_0, x)$

since $|\xi - x_0| \leq |h|$

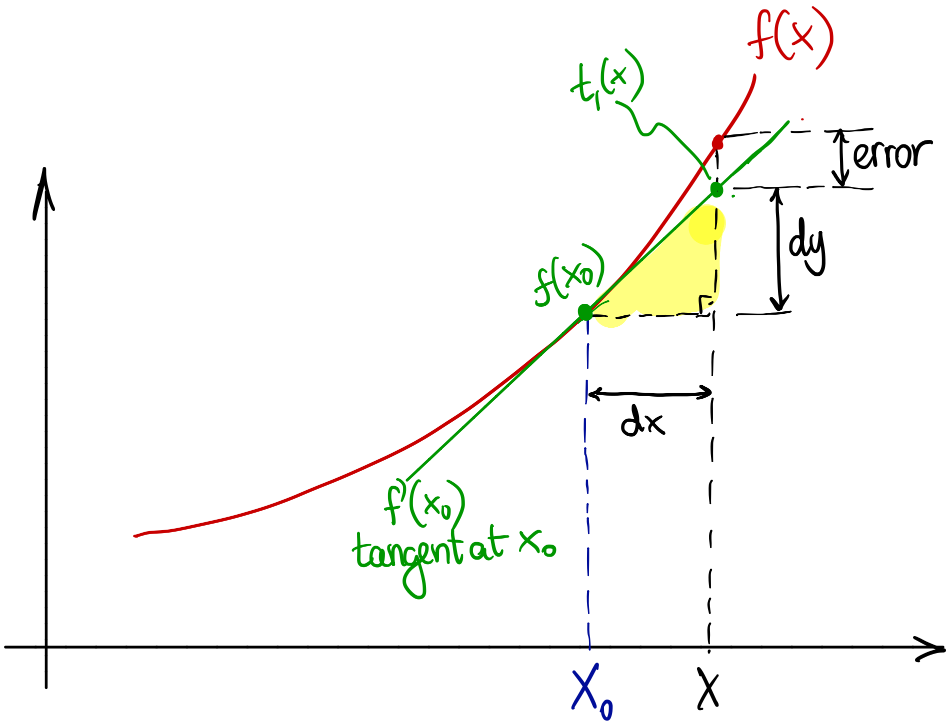
$$|R_n| \leq \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \right|$$

note

$$M = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \right|$$



Graphical representation:



$$f'(x_0) = \frac{dy}{dx} \Rightarrow dy = f'(x_0)(x - x_0)$$

$$t_1(x) = f(x_0) + f'(x_0)(x - x_0) \quad \checkmark$$

$$\text{error} = f(x) - t_1(x) = \text{Remainder}$$

$$\text{error} \le \frac{f''(\xi)(x - x_0)^2}{2!} \qquad \xi \in (x_0, x)$$

$$\boxed{\text{error} = O(h^2)}$$

suppose interval is reduced by half.
what happens to the error?

$$e_1 — h_1^2 \qquad \frac{e_1}{e_2} = 2^2 \qquad \boxed{e_2 = \frac{e_1}{4}}$$

$$e_2 — \left(\frac{h_1}{2}\right)^2$$

Example:

Demo

Given the function

$$f(x) = \frac{1}{(20x - 10)}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

Given the function : $f(x) = \frac{1}{20x - 10}$

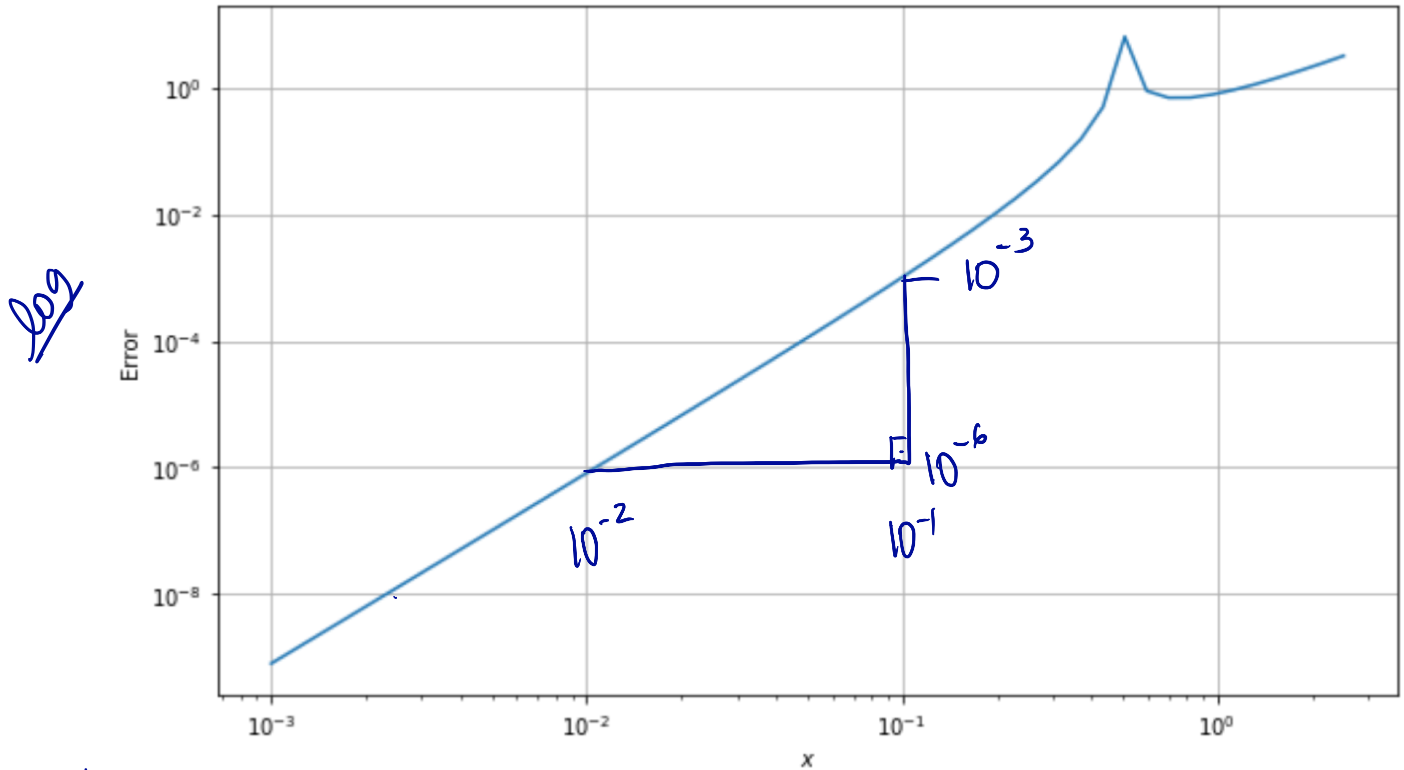
write the Taylor approximation of degree 2 about $x_0 = 0$

$f'(x) = \frac{-1(20)}{(20x-10)^2}$; $f'(0) = \frac{-20}{(-10)^2} = -\frac{1}{5}$

$f''(x) = \frac{+20(20x-10)2(20)}{(20x-10)^4} = \frac{-800}{(20x-10)^3}$ $\quad f''(0) = \frac{-800}{1000} = -4/5$

$$\boxed{t_2(x) = -\frac{1}{10} - \frac{1}{5}x - \frac{1}{2}\left(\frac{4}{5}\right)x^2}$$

$|R_2(x)| \leq \left|\frac{f'''(0)}{3!}x^3\right|$ $\quad$ error $= O(x^3)$



log

$$y = ax^b = \text{error}$$

$$\log(y) = \log(a) + \underbrace{b}_{\text{slope!}} \log(x)$$

$$b = \frac{\log 10^{-3} - \log 10^{-6}}{\log 10^{-1} - \log 10^{-2}} = \frac{-3+6}{-1+2} = \frac{3}{1} \Rightarrow \text{error} = O(x^3) \checkmark$$

Example:

DEMO

Given the function

$$f(x) = \sqrt{-x^2 + 1}$$

Write the Taylor approximation of degree 2 about point $x_o = 0$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

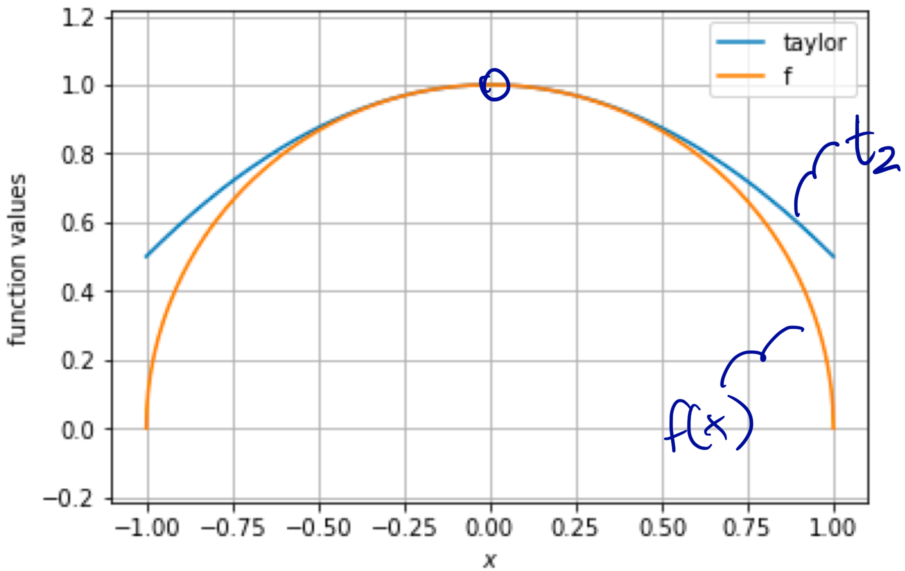
$$f(0) = 1$$

$$f'(x) = \frac{1}{\cancel{2}}(-x^2+1)^{-1/2}(-\cancel{2}x) = -x(1-x^2)^{-1/2} \longrightarrow f'(0) = 0$$

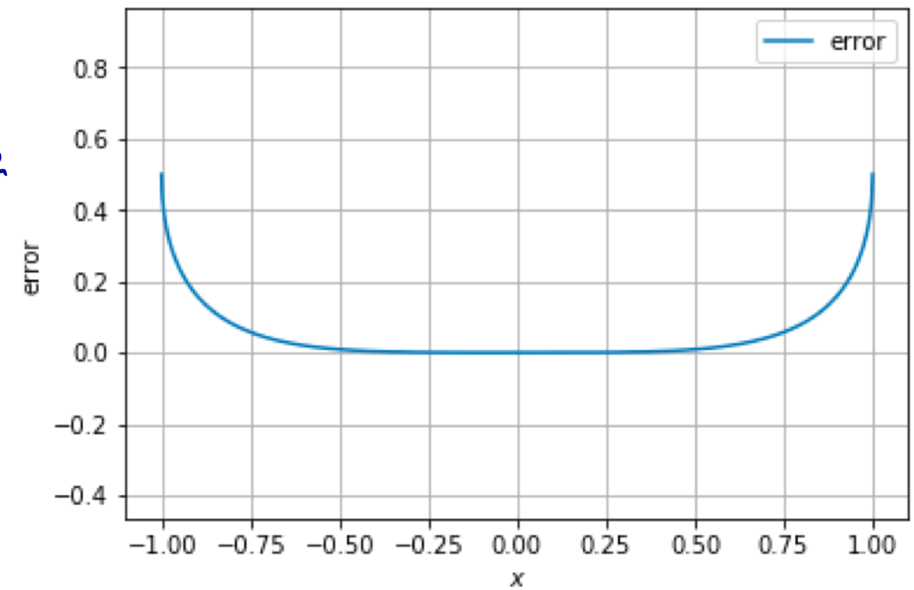
$$f''(x) = -\frac{1}{2}x(1-x^2)^{-3/2}(-2x) - (1-x^2)^{-1/2} \longrightarrow f''(0) = -1$$

$$\hat{f}(x) = 1 - \frac{1}{2}(x)^2 \quad \text{or} \quad \boxed{t_2(x) = 1 - \frac{x^2}{2}}$$

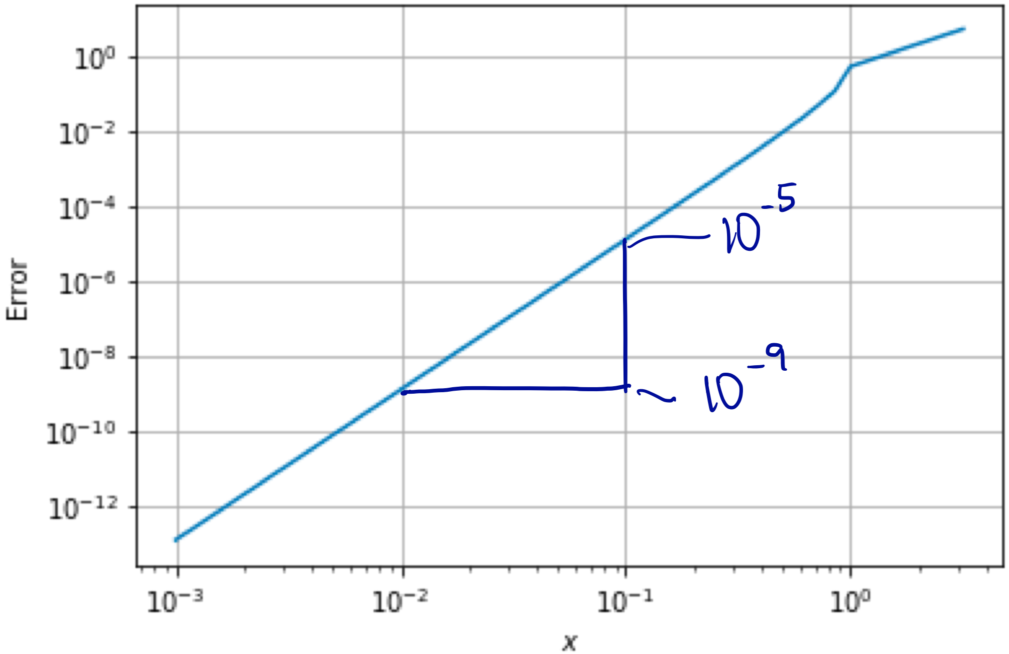
$$f(x) = \sqrt{-x^2 + 1}$$



$$\text{error} = t_2 - f(x)$$



- "good" approximation close to $x_0 = 0$
- error increases when x moves away from x_0
- use log-log plot to better visualize what is happening close to x_0.



$$\text{error} = t_2(x) - f(x)$$

$$f(x) = \sqrt{-x^2+1}$$

$$t_2(x) = 1 - x^2/2$$

$$|R_2| \leq \left| \frac{f'''(\xi)}{3!} h^3 \right| = O(h^3)$$

here $h = x - x_0 = x$

Let's get Big-O of error from the plot!

$$\text{slope} = \frac{\log(10^{-5}) - \log(10^{-9})}{\log(10^{-1}) - \log(10^{-2})} = \frac{-5+9}{-1+2} = 4 \implies \text{error} = O(h^4)$$

what happened here!

$\rightarrow f'''(x) = 0$ hence the next term that is not zero is $f^{IV}(x)$

Example:

DEMO

Error Order for Taylor series

1 point

The series expansion for e^x about 2 is

$$\underbrace{\exp(2)\cdot\left(1+(x-2)+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}}_{t_3(x)}+\ldots\right).$$

If we evaluate e^x using only the first four terms of this expansion (i.e. only terms up to and including $\frac{(x-2)^3}{3!}$), then what is the error in big-O notation?

Choice*

A) $O(x^4)$

B) $O(x^5)$

C) $O(x^3)$

D) $O((x-2)^3)$

E) $O((x-2)^4)$

all are valid options!

this is tightest bound.

$$\text{error} = e^2\left[\frac{(x-2)^4}{4!}+\frac{(x-2)^5}{5!}+\ldots\right]$$

$$\text{error} = O\left((x-2)^4\right)$$

★ $e \leq M(x-2)^4$

as $x\to 2$, e becomes smaller

★ $e \leq M x^4$

as $x \to 2$, $e(x)$ does not show asymptotic behavior

Demo "Taylor of exp(x) about 2"

Finite difference approximation

For a given smooth function $f(x)$, we want to calculate the derivative $f'(x)$ at $x = 1$.

Suppose we don't know how to compute the analytical expression for $f'(x)$, but we have available a code that evaluates the function value:

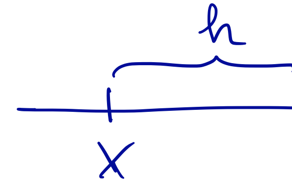
```
def f(x):
    # do stuff here
    feval = ...
    return feval
```

Can we find an approximation for the derivative with the available information? YES!

Calculus $\quad f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$

Can we simply use $\quad f'(x) \approx \frac{f(x+h) - f(x)}{h}$ and make h small? How do we choose h? what is the error?

$f:\mathbb{R}\to\mathbb{R}$ and f is smooth



h: small change (perturbation) to x.

- Taylor Series :

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \dots$$ (just replaced x_0 with x)

$$f(x+h) = f(x) + f'(x)h + O(h^2)$$

$$\frac{f(x+h)-f(x)}{h} = f'(x) + O(h) \longrightarrow \underbrace{f'(x)}_{\text{exact}} = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\substack{\text{approx}\\ \text{(Forward F.D. approx)}}} - \underbrace{O(h)}_{\text{error}}$$

$$\boxed{df(x) = \frac{f(x+h)-f(x)}{h}}$$

$$\boxed{\text{error due to FD approx} \Rightarrow e_t = O(h)}$$

Demo: Finite Difference

$f(x) = e^x - 2$

We want to obtain an approximation for $f'(1)$

$dfexact = e^x$

$$dfapprox = \frac{\overbrace{e^{x+h} - 2}^{f(x+h)} - \overbrace{(e^x - 2)}^{f(x)}}{h}$$

$error(h) = abs(dfexact - dfapprox)$

$error < \left| f''(\xi)\frac{h}{2} \right|$

truncation error

Demo: Finite Difference

Demo

$$f(x) = e^x - 2$$

We want to obtain an approximation for $f'(1)$

$$dfexact = e^x$$

$$dfapprox = \frac{e^{x+h} - 2 - (e^x - 2)}{h}$$

$$error(h) = abs(dfexact - dfapprox)$$

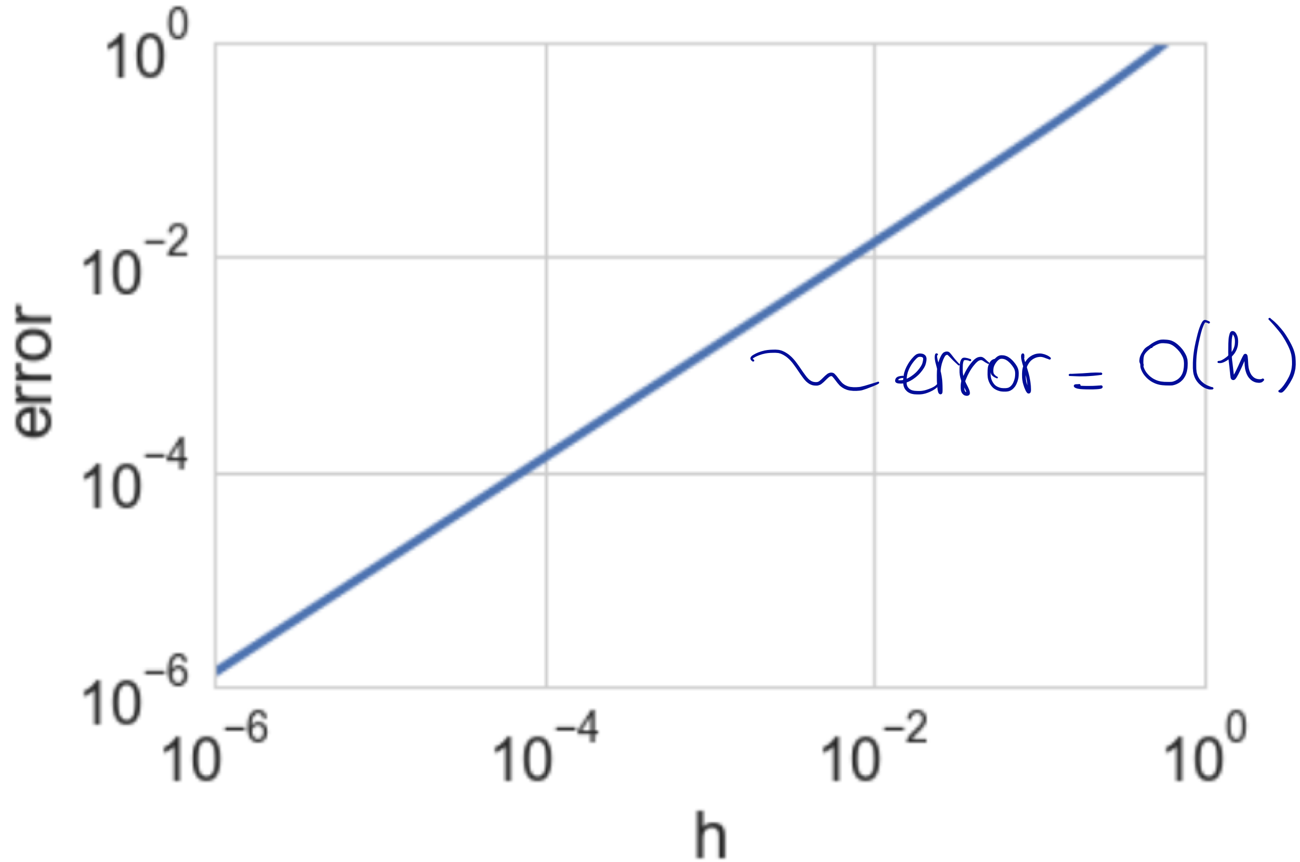
↳ absolute error!

$$error < \left| f''(\xi)\frac{h}{2} \right|$$

truncation error

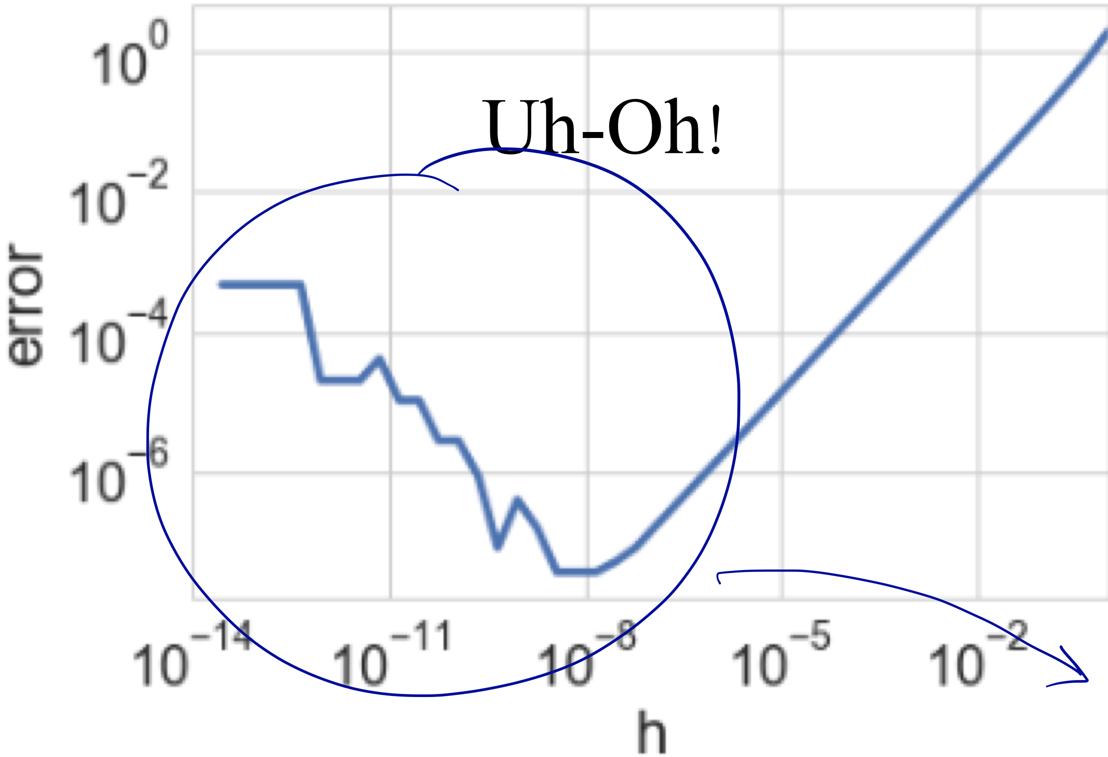
h	$error$
1.000000E+00	1.952492E+00
5.000000E-01	8.085327E-01
2.500000E-01	3.699627E-01
1.250000E-01	1.771983E-01
6.250000E-02	8.674402E-02
3.125000E-02	4.291906E-02
1.562500E-02	2.134762E-02
7.812500E-03	1.064599E-02
3.906250E-03	5.316064E-03
1.953125E-03	2.656301E-03
9.765625E-04	1.327718E-03
4.882812E-04	6.637511E-04
2.441406E-04	3.318485E-04
1.220703E-04	1.659175E-04
6.103516E-05	8.295707E-05
3.051758E-05	4.147811E-05
1.525879E-05	2.073897E-05
7.629395E-06	1.036945E-05
3.814697E-06	5.184779E-06
1.907349E-06	2.592443E-06

Demo: Finite Difference



$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Should we just keep decreasing the perturbation h, in order to approach the limit $h \to 0$ and obtain a better approximation for the derivative?



What happened here?

$$f(x) = e^x - 2$$

$$f'(x) = e^x \to f'(1) \approx 2.7$$

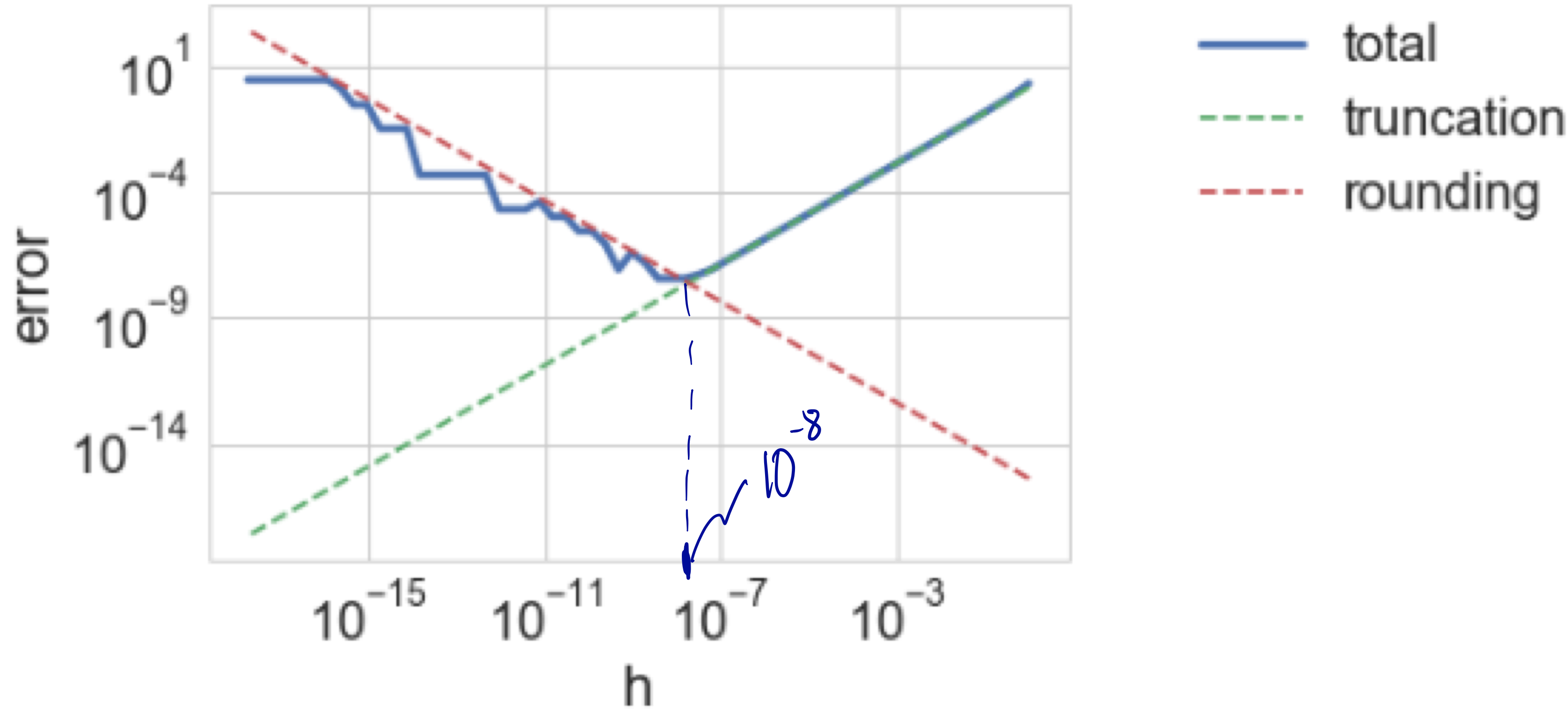
$$f'(1) = \lim_{h \to 0} \left(\frac{f(1+h) - f(1)}{h} \right)$$

Rounding error starts to creep up!

1) very small h ($h \leq 10^{-8}$)
$f(1+h) - f(1) \longrightarrow$ starts increasing
losing precision due to cancelation!

2) even smaller ($h \leq 10^{-16}$) $\longrightarrow f(1+h) = f(1) \longrightarrow df = 0$

↳ machine epsilon (because we are adding h to 1. If another FP number, than this would be the gap.



$$h = \frac{\epsilon}{h} \Rightarrow h^2 = \epsilon$$

$$h = \sqrt{\epsilon}$$

Truncation error: $error \sim M\frac{h}{2}$

$$h = \sqrt{10^{-16}}$$

Rounding error: $error \sim \frac{2\epsilon}{h}$

$$\frac{f(1+h) - f(1)}{h}$$

$$h = 10^{-8}$$