Data Structures and Algorithms
Probability in Computer Science

CS 225
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Fundamentals of Probability (Correction)

Linearity of Expectation: For any two random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

$$= \sum_x \sum_y (x+y) Pr\{X=x, Y=y\}$$

$$= \sum_x x \sum_y Pr\{X=x, Y=y\} + \sum_y y \sum_x Pr\{X=x, Y=y\}$$

$$= \sum_x x \cdot Pr\{X=x\} + \sum_y y \cdot Pr\{Y=y\}$$

does not depend on independence!

Randomization in Algorithms

1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed tim

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \log n)$

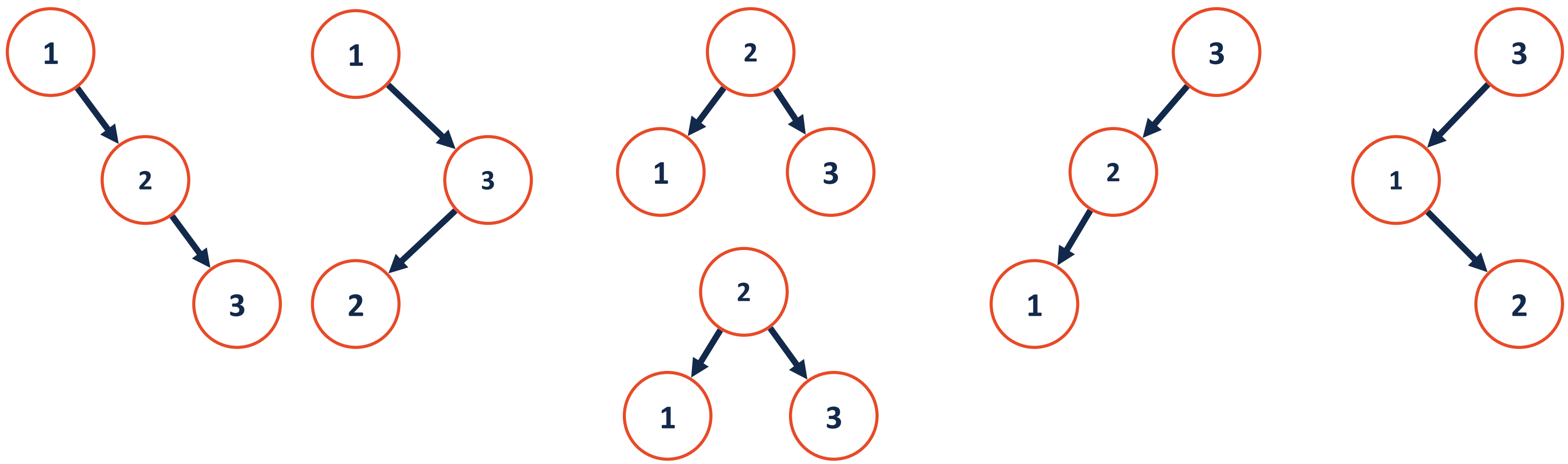
N=0:

N=1:

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

N=3:



Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Let $0 \leq i \leq n-1$ be the number of nodes in the left subtree.

Then for a fixed i, $S(n) = (n-1) + S(i) + S(n-i-1)$

Average-Case Analysis: BST

Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

$$S(n) = (n-1) + \frac{1}{n}\sum_{i=0}^{n-1} S(i) + S(n-i-1)$$

Average-Case Analysis: BST

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} S(i)$$

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} (ci \ln i)$$

$$S(n) \leq (n-1) + \frac{2}{n}\int_{1}^{n} (cx \ln x)dx$$

$$S(n) \leq (n-1) + \frac{2}{n}\left(\frac{cn^2}{2}\ln n - \frac{cn^2}{4} + \frac{c}{4}\right) \approx cn \ln n$$

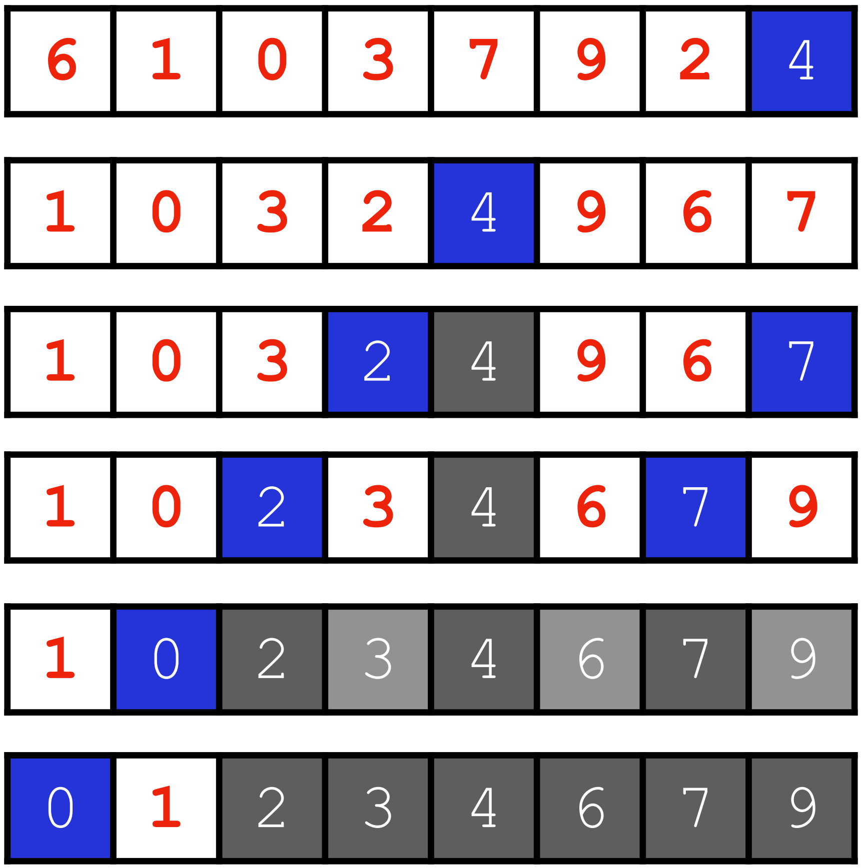
Average-Case Analysis: BST

Summary: All operations are on average $O(logn)$

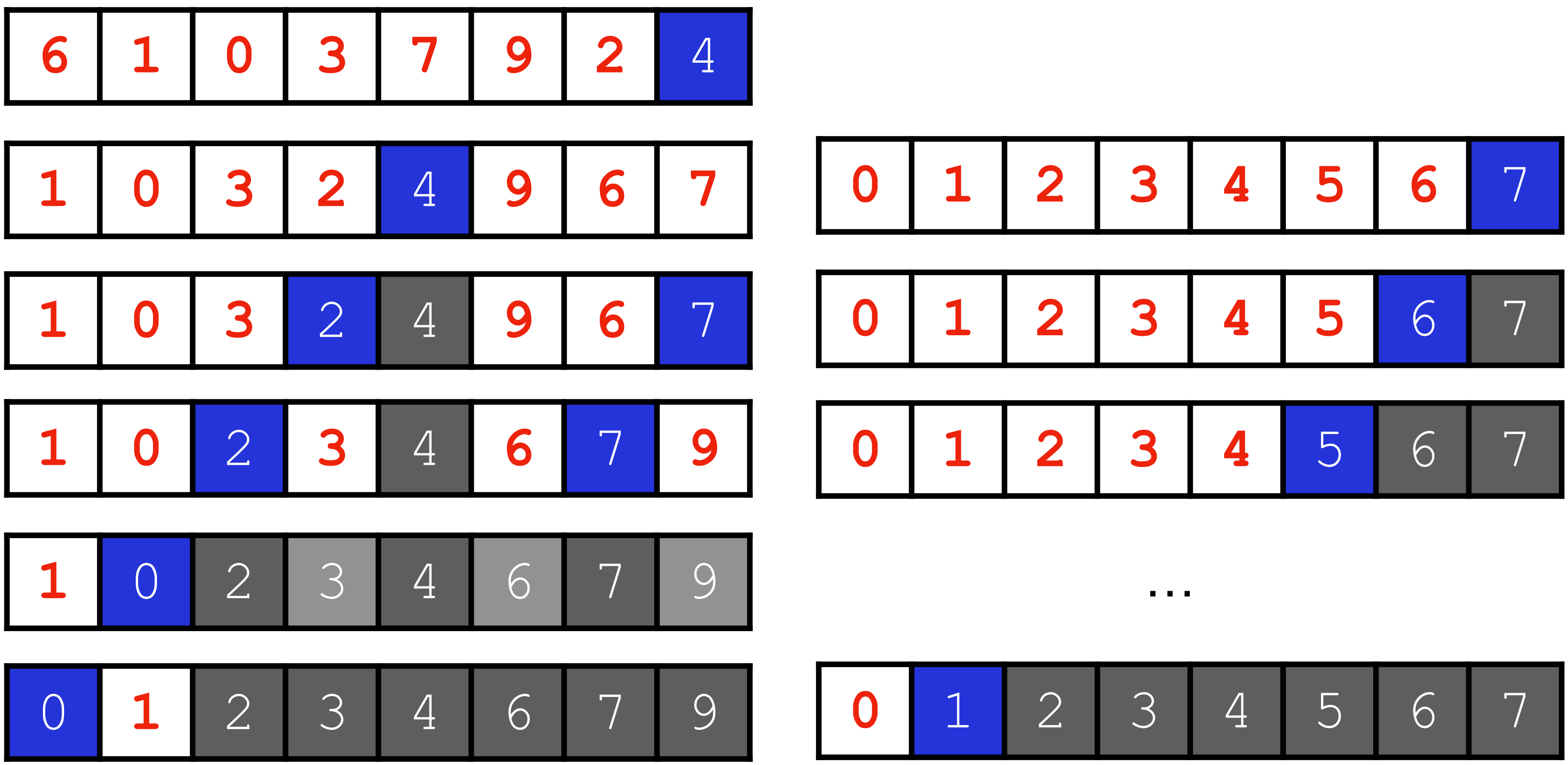
Randomness:

Assumptions:

Expectation Analysis: Randomized Quicksort



Expectation Analysis: Randomized Quicksort



Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected comparisons is $O(n \log n\, n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \{ \begin{matrix} 1 \text{if } i\text{th object compared to } j\text{th} \\ 0 \text{ if } i\text{th object not compared to } j\text{th} \end{matrix}$$

Then…

Key Ideas

1. Never compare X_i with X_i

2. Never compare X_i and X_j more than once

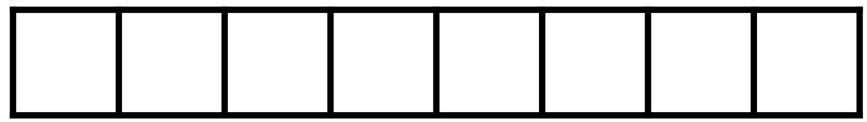
Expectation Analysis: Randomized Quicksort

Claim: $E[X_{ij}] = \frac{2}{j-i+1}$

Base Case: (N=2)

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \frac{2}{j-i+1}$ **Induction:** Assume true for all inputs of $< n$



Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$E[X_{ij}] = \frac{2}{j - i + 1}$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$$

$$E[X_{ij}] = \frac{2}{j - i + 1}$$

$$E[X] = \sum_{i=1}^{n} 2(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n - i + 1})$$

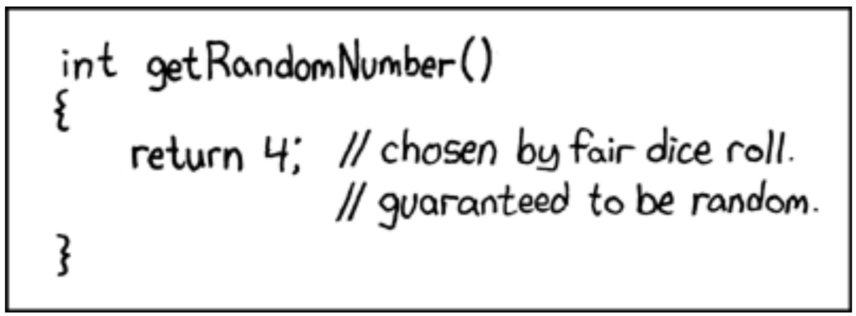
$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2n\, ln\, n\, n$$

Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(nlogn)$ regardless of input

Randomness:

Assumptions:


int getRandomNumber()
{
return 4; // chosen by fair dice roll.
// guaranteed to be random.
}

Probabilistic Accuracy: Fermat primality test

Pick a random a in the range $[2, p-2]$

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 (mod\,p)$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 (mod\,n)$

Probabilistic Accuracy: Fermat primality test

	$a^{p-1} \equiv 1 (mod\, p)$	$a^{p-1} \not\equiv 1 (mod\, p)$
p is prime		
p is not prime		

Probabilistic Accuracy: Fermat primality test

Let's assume $\alpha = .5$

First trial: $a = a_0$ and prime test returns 'prime!'

Second trial: $a = a_1$ and prime test returns 'prime!'

Third trial: $a = a_2$ and prime test returns 'not prime!'

Is our number prime?

What is our **false positive** probability? Our **false negative** probability?

Probabilistic Accuracy: Fermat primality test

Summary: Randomized algorithms can also have fixed (or bounded) runtimes at the cost of probabilistic accuracy.

Randomness:

Assumptions:

Types of randomized algorithms

A **Las Vegas** algorithm is a randomized algorithm which will always give correct answer if run enough times but has no fixed runtime.

A **Monte Carlo** algorithm is a randomized algorithm which will run a fixed number of iterations and may give the correct answer.

Next Class: Randomized Data Structures

Sometimes a data structure can be **too ordered / too structured**

Randomized data structures rely on **expected** performance

Randomized data structures ‘cheat’ tradeoffs!