Data Structures and Algorithms
Probability in Computer Science

CS 225
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Learning Objectives

Formalize the concept of randomized algorithms

Review fundamentals of probability in computing

Distinguish the three main types of ‘random’ in computer science

Randomized Algorithms

A **randomized algorithm** is one which uses a source of randomness somewhere in its implementation.

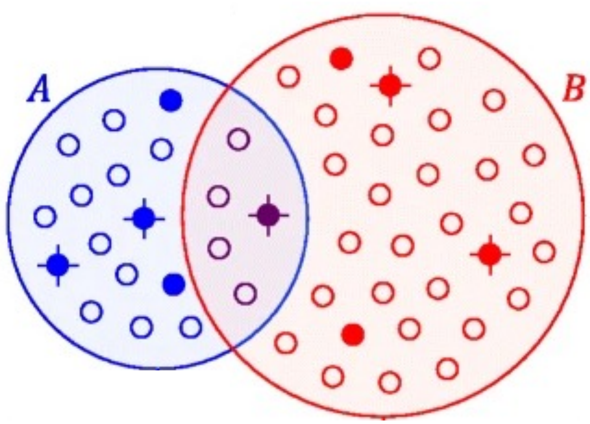
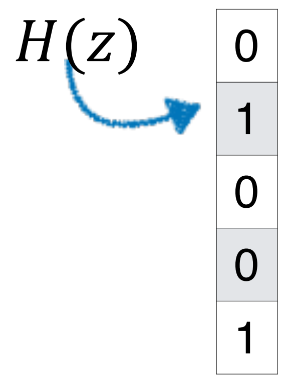
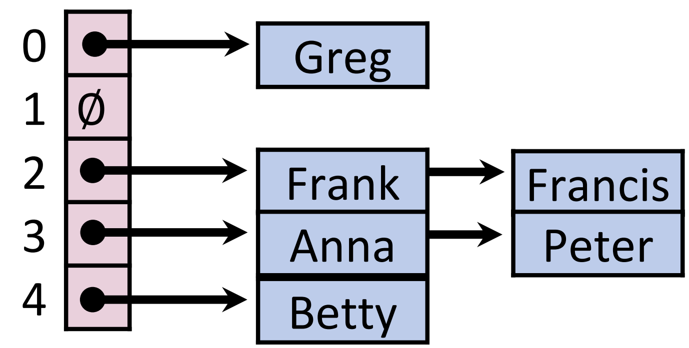


Figure from Ondov et al 2016





$H(x)$	0	2	1	0	0	4	0	2	0	6
$H(y)$	1	0	2	3	1	0	3	4	0	1
$H(z)$	2	1	0	2	0	1	0	0	7	2

Quick Primes with Fermat's Primality Test

If p is prime and a is not divisible by p, then $a^{p-1} \equiv 1 (mod\, p)$

But… ***sometimes*** if n is composite and $a^{n-1} \equiv 1 (mod\, n)$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice.

The **sample space** Ω is the set of all possible outcomes.

An **event** $E \subseteq \Omega$ is any subset.

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

A **random variable** is a function from events to numeric values.

The **expectation** of a (discrete) random variable is:

$$E[X] = \sum_{x \in \Omega} Pr\{X = x\} \cdot x$$

Fundamentals of Probability

Imagine you roll a pair of six-sided dice. What is the expected value?

$$E[X + Y] = ?$$

Fundamentals of Probability

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Linearity of Expectation: For any two random variables X and Y,

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$$E[X+Y] = E[X] + E[Y]$$

$$] = \sum_x \sum_y (x+y) Pr\{X = x, Y = y\}$$

$$] = \sum_x x \sum_y Pr\{X = x, Y = y\} + \sum_y y \sum_x Pr\{X = x, Y = y\}$$

$$] = \sum_x x \cdot Pr\{X = x\} + \sum_y y \cdot Pr\{Y = y\}$$

Randomization in Algorithms

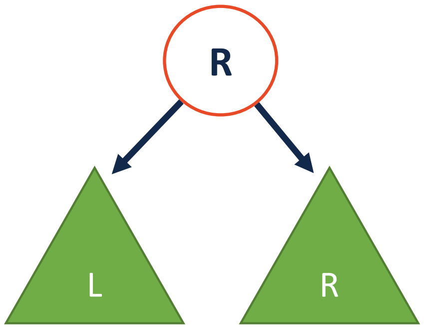
1. Assume input data is random to estimate average-case performance

2. Use randomness inside algorithm to estimate expected running time

3. Use randomness inside algorithm to approximate solution in fixed time

Average-Case Analysis: BST

Smallest Largest



Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Claim: $S(n)$ is $O(n \log n)$

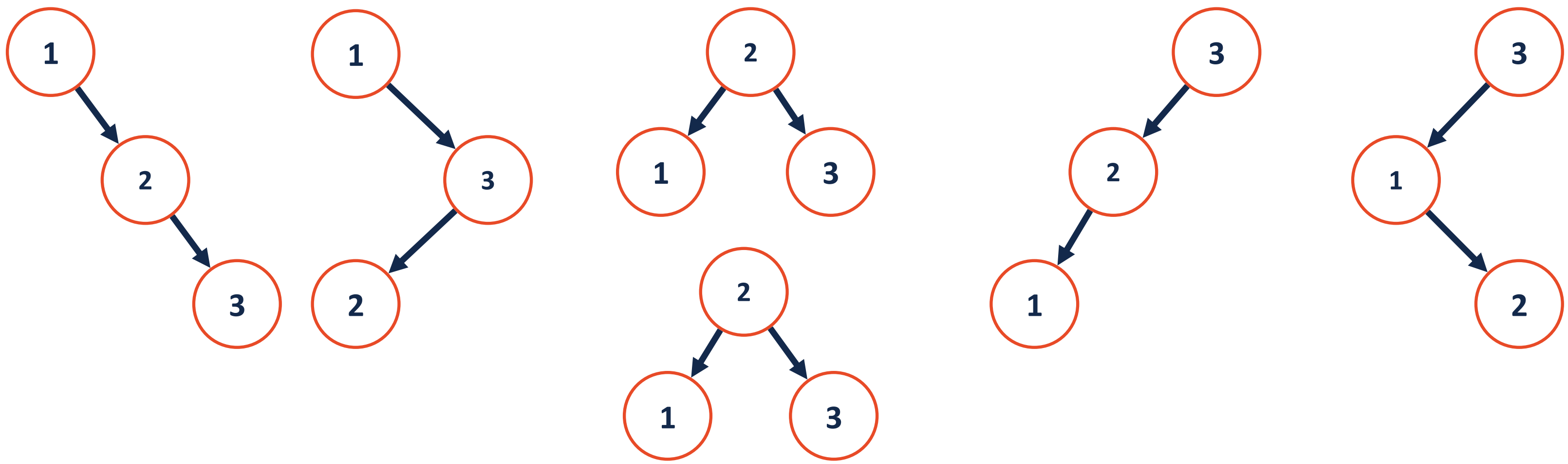
N=0:

N=1:

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

N=3:



Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

$\text{IH for all } 0 \leq k < n \;\; S(k)$ is $O(k \log k)$

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Let $0 \leq i \leq n-1$ be the number of nodes in the left subtree.

Then for a fixed i, $S(n) = (n-1) + S(i) + S(n-i-1)$

Average-Case Analysis: BST

Let $S(n)$ be the **average** total internal path length **over all BSTs** that can be constructed by uniform random insertion of n objects

$$S(n) = (n-1) + \frac{1}{n} \sum_{i=1}^{n-1} S(i) + S(n-i-1)$$

Average-Case Analysis: BST

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} S(i)$$

$$S(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} (ci \ln i)$$

$$S(n) \leq (n-1) + \frac{2}{n}\int_{1}^{n} (cx \ln x)dx$$

$$S(n) \leq (n-1) + \frac{2}{n}\left(\frac{cn^2}{2}\ln n - \frac{cn^2}{4} + \frac{c}{4}\right) \approx cn \ln n$$

Average-Case Analysis: BST

Let $S(n)$ be the average **total internal path length** over all BSTs that can be constructed by uniform random insertion of n objects

Since $S(n)$ is $O(n \log n)$, if we assume we are randomly choosing a node to insert, find, or delete* then each operation takes:

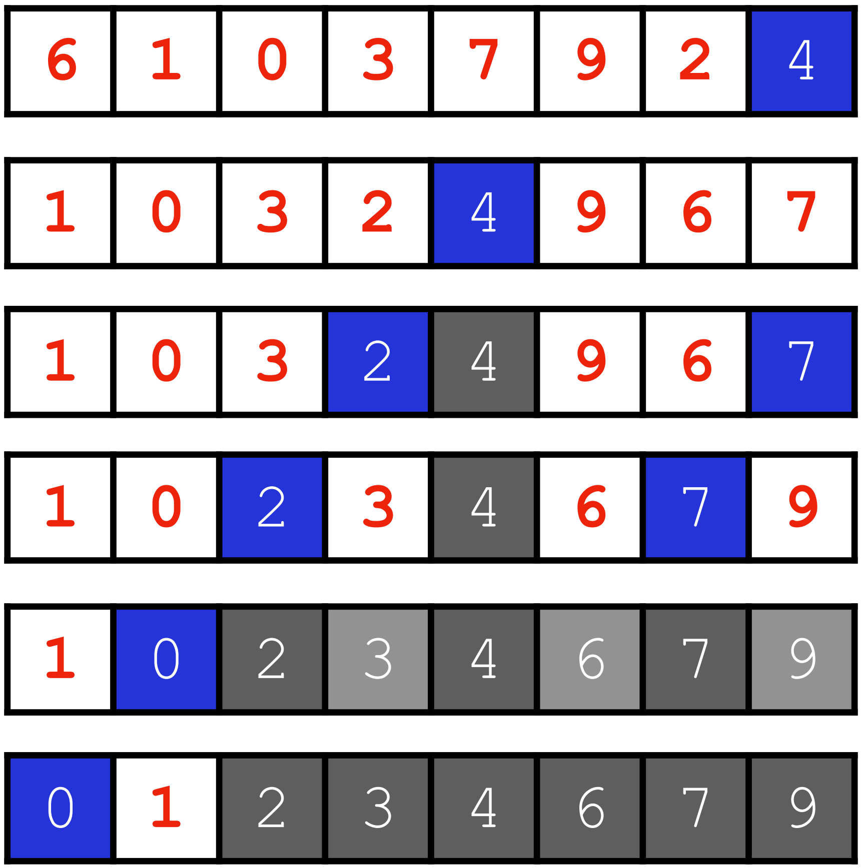
Average-Case Analysis: BST

Summary: All operations are on average $O(logn)$

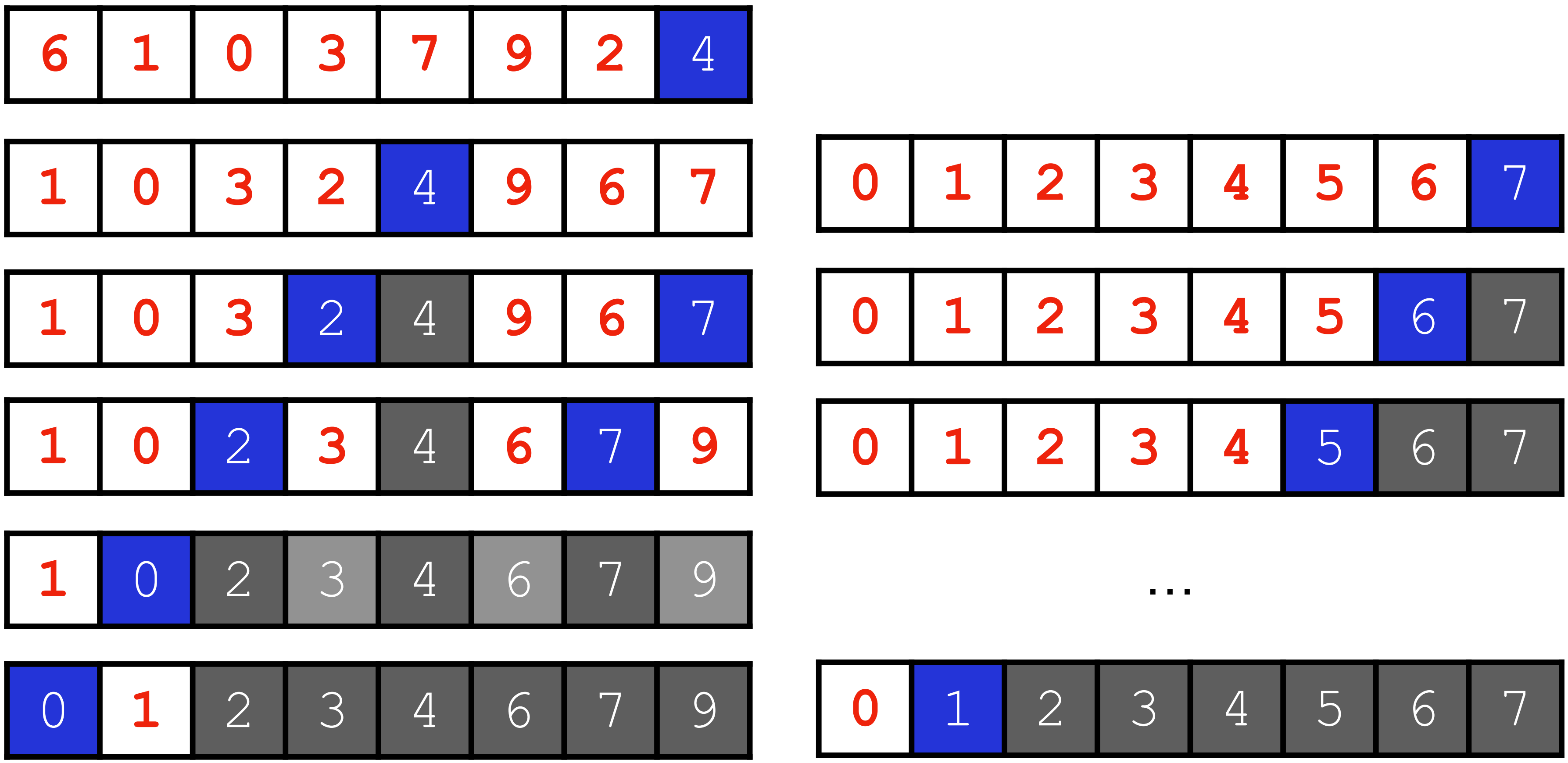
Randomness:

Assumptions:

Expectation Analysis: Randomized Quicksort



Expectation Analysis: Randomized Quicksort



Expectation Analysis: Randomized Quicksort

In **randomized quicksort**, the selection of the pivot is random.

Claim: The expected time is $O(n \log n)$ ***for any input!***

Expectation Analysis: Randomized Quicksort

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Claim: The expected time is $O(n \log n)$ ***for any input!***

Let X be the total comparisons and X_{ij} be an **indicator variable**:

$$X_{ij} = \begin{cases} 1 \text{ if } ith \text{ object compared to } jth \\ 0 \text{ if } i \text{ object not compared to } jth \end{cases}$$

Then…

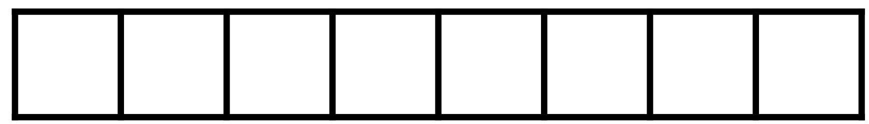
Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \frac{2}{j-i+1}$.

Base Case: (N=2)

Expectation Analysis: Randomized Quicksort

Claim: $E[X_{i,j}] = \frac{2}{j-i+1}$ **Induction:** Assume true for all inputs of $< n$



Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} E[X_{ij}]$$

$$E[X_{ij}] = \frac{2}{j - i + 1}$$

Expectation Analysis: Randomized Quicksort

$$E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] \quad E[X_{ij}] = \frac{2}{j-i+1}$$

$$E[X] = \sum_{i=1}^{n} 2(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-i+1})$$

$$E[X] = \sum_{i=1}^{n} 2(H_{n-1} - 1) \leq 2n \cdot H_n \leq 2nlnn$$

Expectation Analysis: Randomized Quicksort

Summary: Randomized quick sort is $O(nlogn)$ regardless of input

Randomness:

Assumptions: