Data Structures

MST Part 3 and Single Source Shortest Path

CS 225

December 1, 2023

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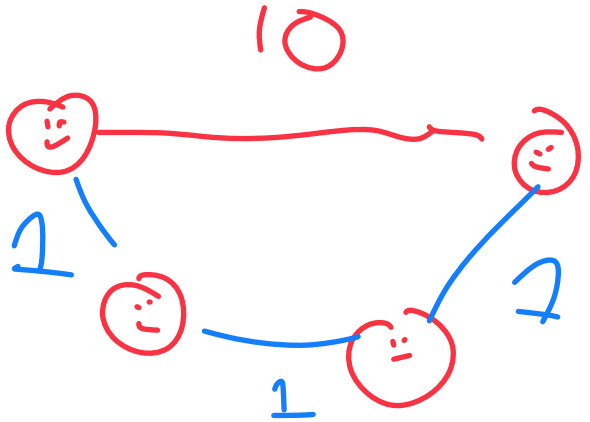


UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Department of Computer Science

Staff Hiring (CS 225 / CS 277)

Both have applications on https://opportunities.cs.illinois.edu/

Both have significant opportunities for dev work

If you regularly interact with course staff, ask them for a rec!

In your application, include things that make you stand out

Learning Objectives

Compare Kruskal and Prim MST Algorithms

Introduce Single-Source Shortest Path Problem

Discuss Dijkstra's Algorithm

All path Shortest path

Kruskal's Algorithm

Priority Queue:	Total Running Time
Heap	O(n + m + m log(n)) *
Sorted Array	O(n + m log(n) + m)

Construction

get / remove Min

1) Disjoint Set + Priority Queue

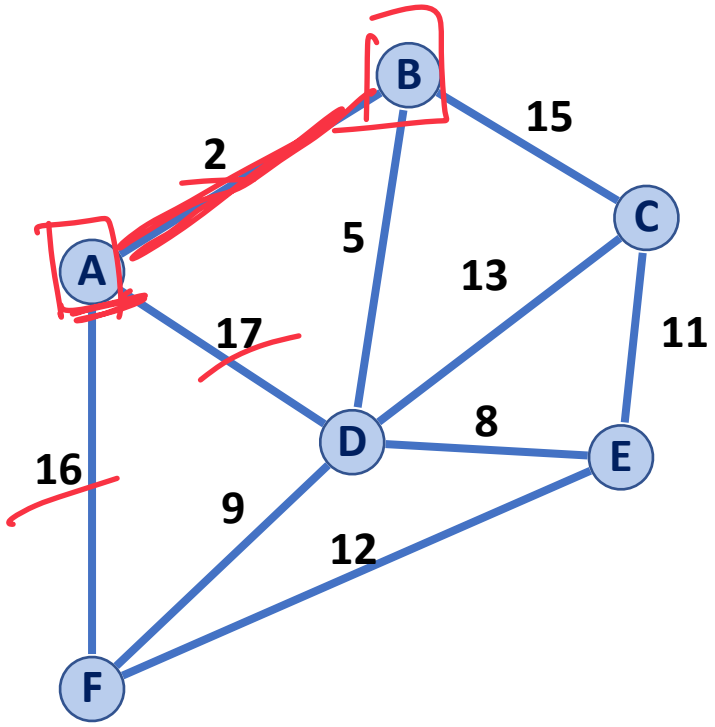
* Destructive

Sorted array persist

```
1   KruskalMST(G):
2     DisjointSets forest
3     foreach (Vertex v : G):
4       forest.makeSet(v)
5
6     PriorityQueue Q     // min edge weight
7     Q.buildFromGraph(G)
8
9     Graph T = (V, {})
10
11    while |T.edges()| < n-1:
12      Vertex (u, v) = Q.removeMin()
13      if forest.find(u) != forest.find(v):
14         T.addEdge(u, v)
15         forest.union( forest.find(u),
16                       forest.find(v) )
17
18    return T
19
```

O(n)

Prim's Algorithm



A	B	C	D	E	F
0, —	2, A	∞, --	17, A	∞, --	16, A

```
1  PrimMST(G, s):
2    Input: G, Graph;
3           s, vertex in G, starting vertex
4    Output: T, a minimum spanning tree (MST) of G
5
6    foreach (Vertex v : G.vertices()):
7      d[v] = +inf
8      p[v] = NULL
9    d[s] = 0
10
11   PriorityQueue Q   // min distance, defined by d[v]
12   Q.buildHeap(G.vertices())
13   Graph T           // "labeled set"
14
15   repeat n times:
16     Vertex m = Q.removeMin()
17     T.add(m)
18     foreach (Vertex v : neighbors of m not in T):
19       if cost(v, m) < d[v]:
20         d[v] = cost(v, m)
21         p[v] = m
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23   return T
```
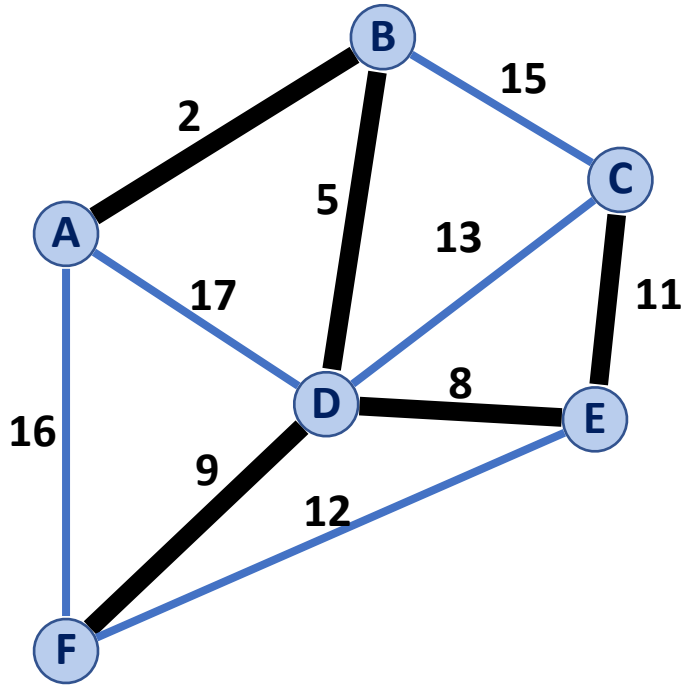

Prim's Algorithm



A	B	C	D	E	F
0, —	2, A	11, C	5, B	8, D	9, D

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7-10: O(n)

13: ???

17: ???

19-22: O(m * ???)

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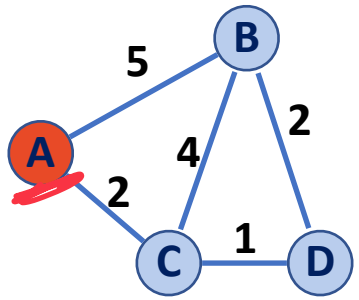
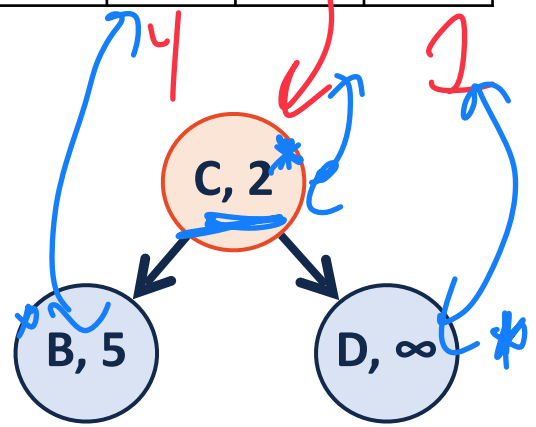
```

A	B	C	D
0	5	2	∞



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```

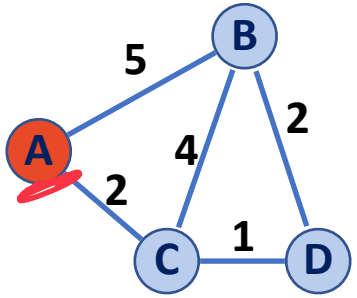
# of heapify up == number of changed costs

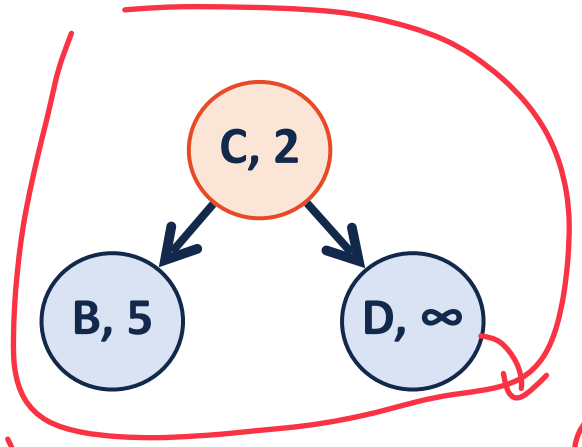
n times neighbors

$\sum_v \deg(v) = 2m$

	Adj. Matrix	Adj. List
Heap	O(n) + O(n log n) + O(n^2) + m log(n)	O(n) + O(n log n) + O(m) + m log(n)

A	B	C	D
0	5	2	∞





1) Remove min

2) Update dist

B,5 → C,2

B,4 → D,7

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23
```

map?

Dist also in heap!

How many times this called?

1) In dist array
2) In heap

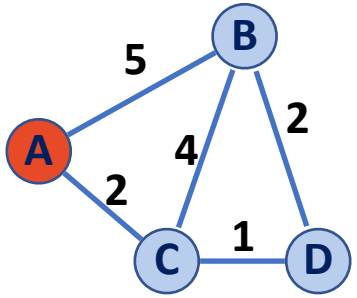
Store pointers not values

// ↳ heapify up()

update one & heapify up()

	Adj. Matrix	Adj. List
Heap	O(n) + _____ + O(n^2) + _____	O(n) + _____ + O(m) + _____

(A, 0)
(D, ∞)
(C, 2)
(B, 5)

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```

getting neighbors is n^2

m

O(n)

O(n²)

O(1)

Why not $m^2 \log n$

Priority queue	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
Unsorted Array	$O(n^2)$	$O(n^2)$

Prim's Algorithm

Sparse Graph: $O(n) \le O(m) \le O(n^2)$

↳ $m \sim n$ → $O(n \log n)$

↳ adj. list heap is best

Dense Graph: $m \sim n^2$

↳ unsorted array is better than heap on dense graph

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	Adj. Matrix	Adj. List
Heap	$O(n^2 + m \lg(n))$	$O(n \lg(n) + m \lg(n))$
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$n^2 \log n$

MST Algorithm Runtime:

Kruskal's Algorithm:
O(n + m log (n))

Prim's Algorithm:
O(n log(n) + m log (n))

Sparse Graph: $m \sim n$

$n + n \log n$

$n \log n + n \log n$

Dense Graph: $m \sim n^2$

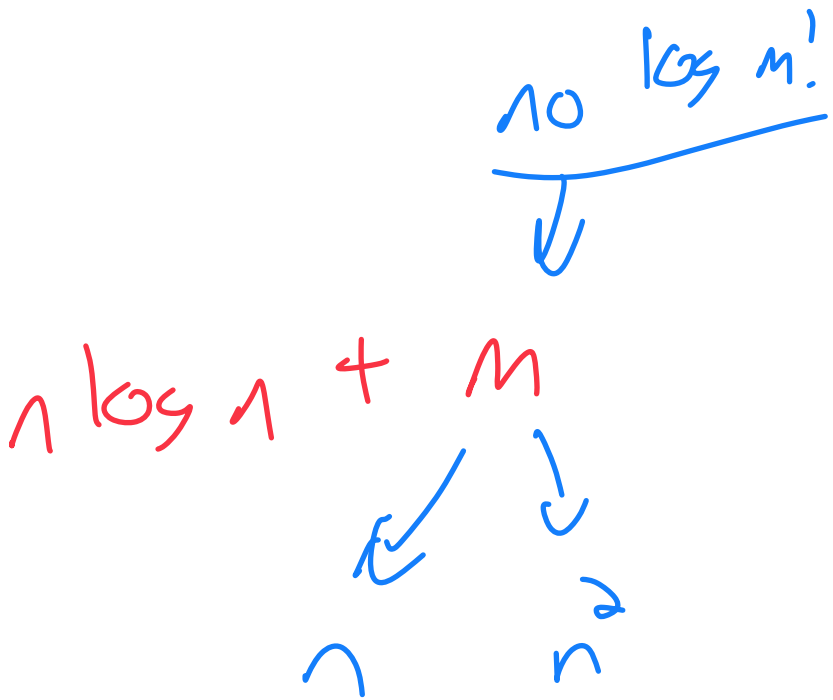
$n + n^2 \log n$

$n \log n + n^2 \log n$

Suppose I have a new heap:

	Binary Heap	Fibonacci Heap
Remove Min	O(lg(n))	O(lg(n))
Decrease Key	O(lg(n))	O(1)*

What's the updated running time?

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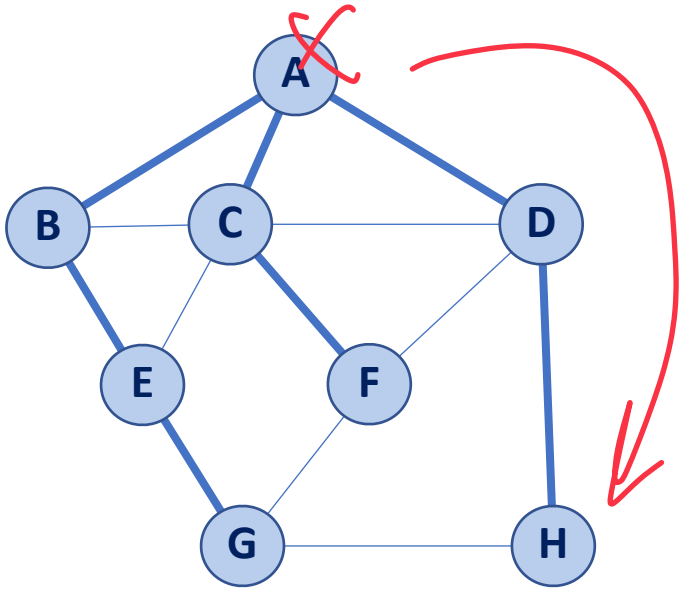
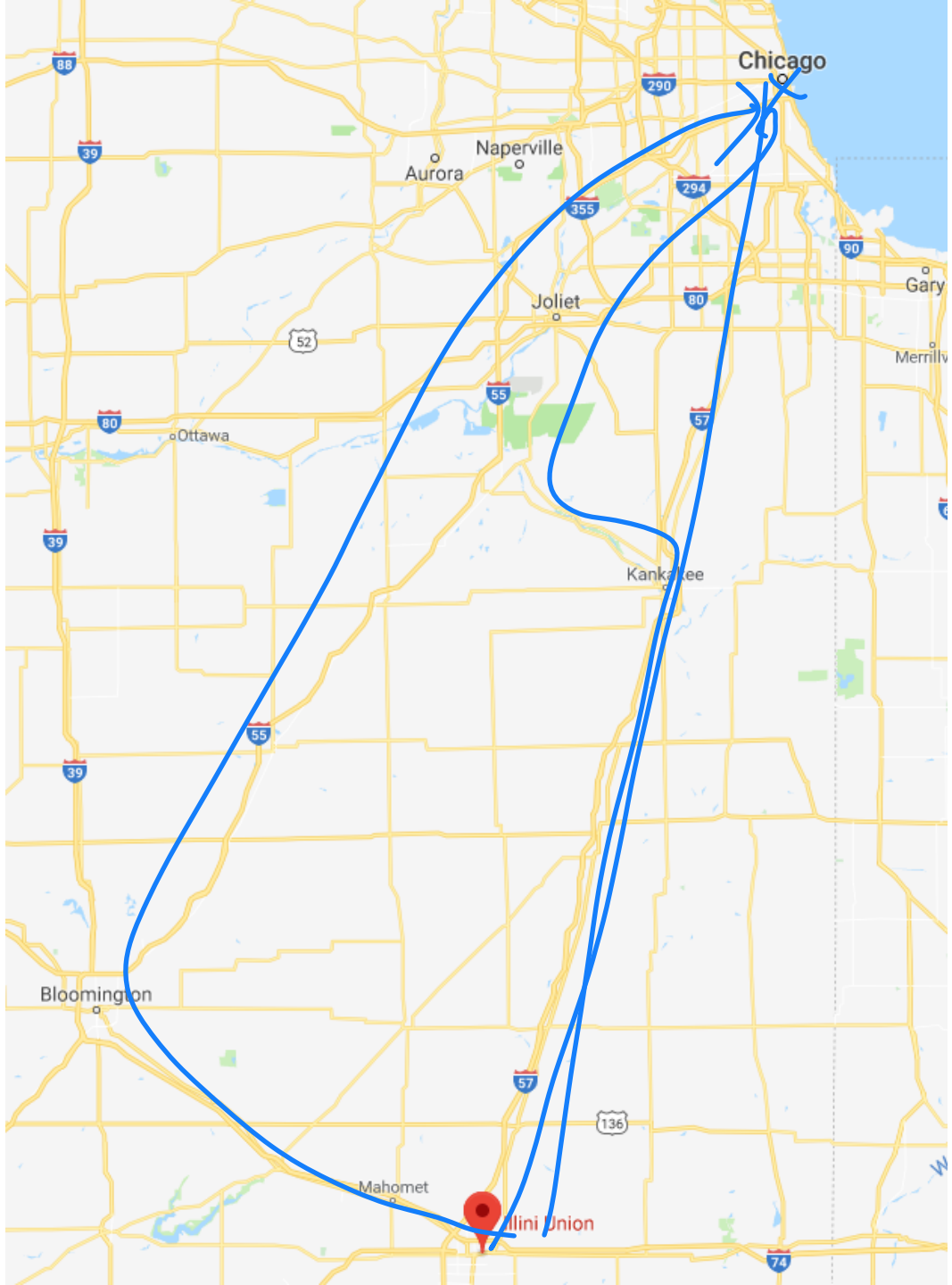
O(log n) vs O(1)

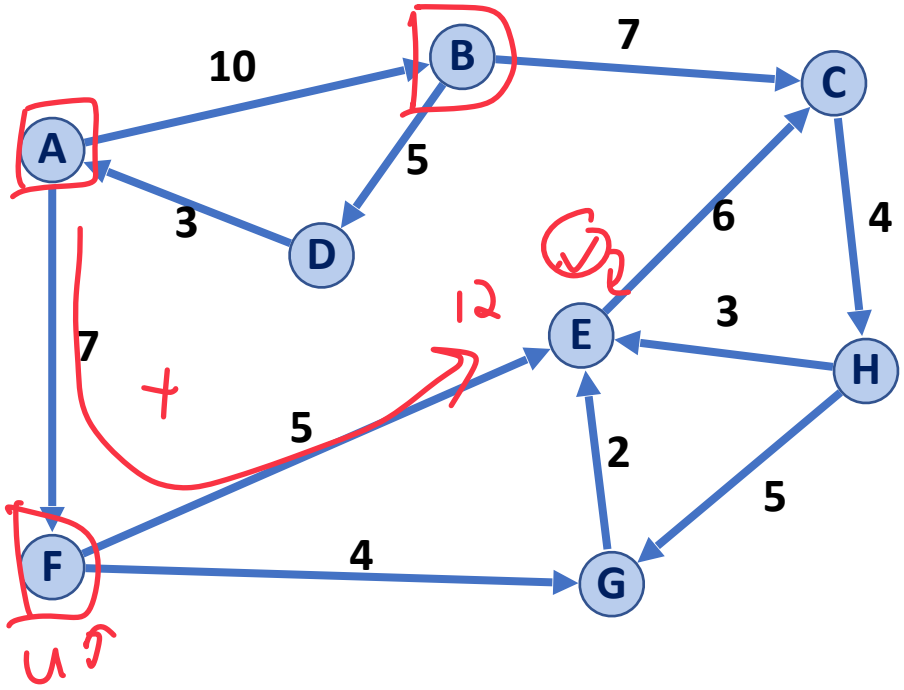
Shortest Path





Dijkstra's Algorithm (SSSP)

Prim but ....

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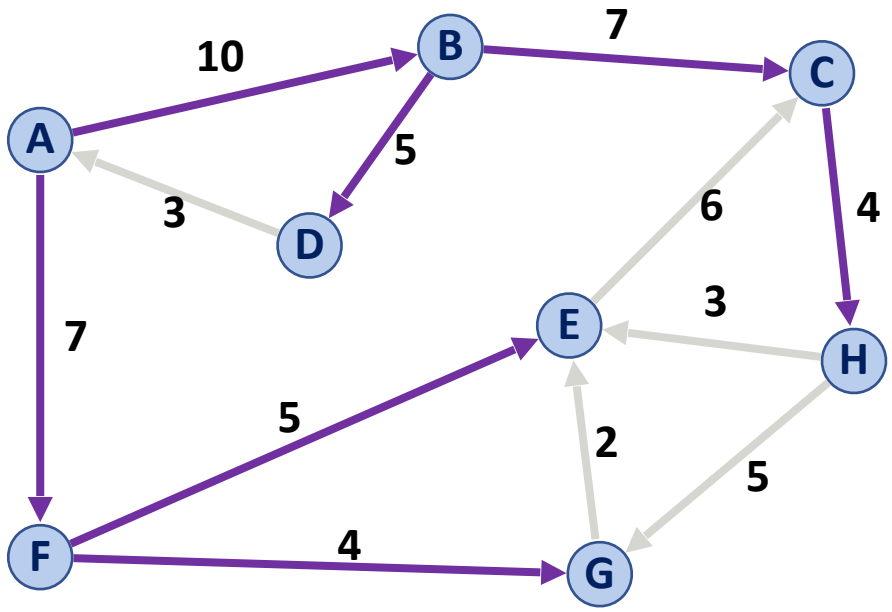
```

A	B	C	D	E	F	G	H
--	A	B	B	F	A	F	
0	10	17	15	12	7	11	

Dijkstra's Algorithm (SSSP)

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```

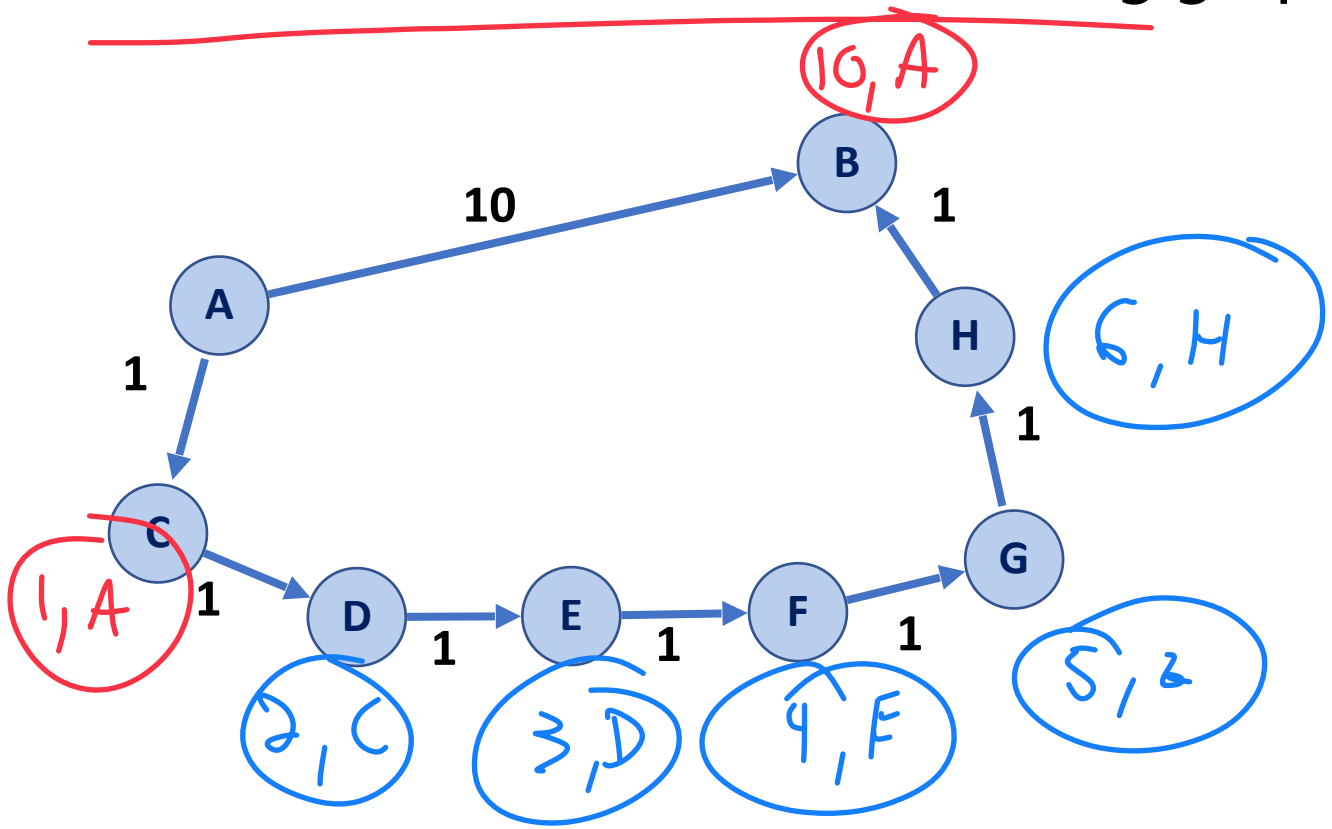
Identifying next smallest tot! dist

A	B	C	D	E	F	G	H
--	A	B	B	F	A	F	C
0	10	17	15	12	7	11	21

Not Greedy!

Dijkstra's Algorithm (SSSP)

When we will visit B in the following graph?

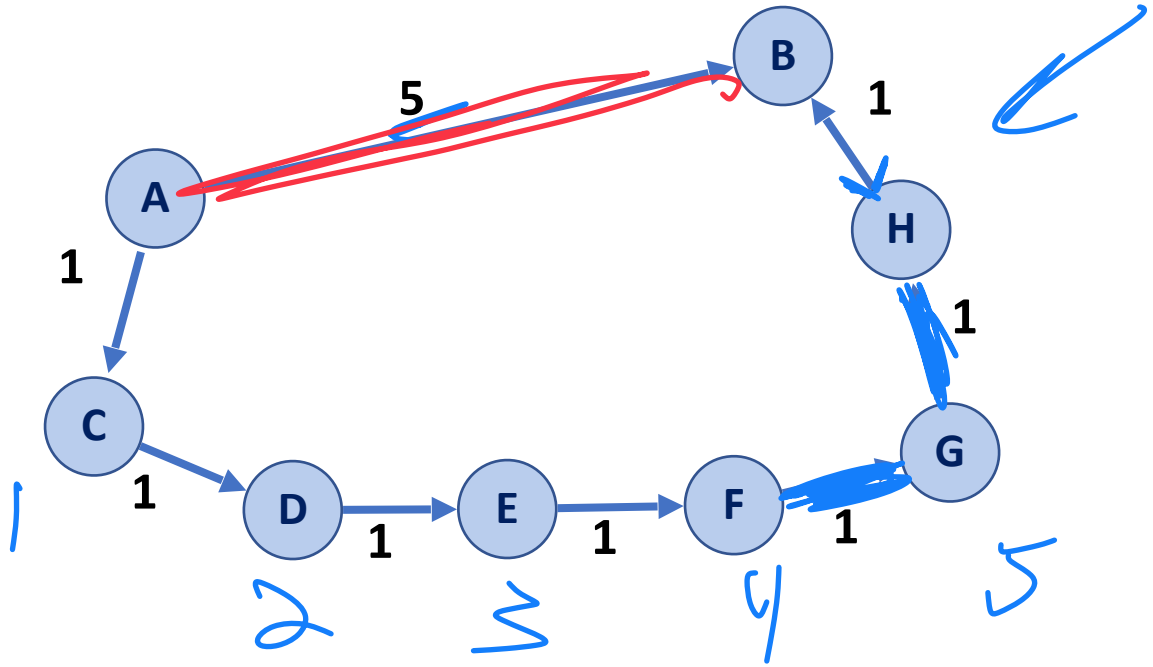


Partition property
↳ min dist btw sets is in my MST

We will visit next smallest global dist using our heap

Dijkstra's Algorithm (SSSP)

When we will visit B in the following graph?



H : 6, B
6, G

Intervals of Priority Queue which one we take

Dijkstra's Algorithm (SSSP)

Running time

What is the running time of Dijkstra's Algorithm?

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