Disjoint Sets

Key Ideas:
- Each element exists in exactly one set.
- Every set is an equitant representation.
  - Mathematically: $4 \in [0]_R \implies 8 \in [0]_R$
  - Programmatically: `find(4) == find(8)`
Disjoint Sets ADT

• Maintain a collection \( S = \{s_0, s_1, \ldots s_k\} \)

• Each set has a representative member.

• API:  
  ```
  void makeSet(const T & t);
  void union(const T & k1, const T & k2);
  T & find(const T & k);
  ```
Implementation #1

Find(k):

Union(k1, k2):
Implementation #2

• We will continue to use an array where the index is the key

• The value of the array is:
  • -1, if we have found the representative element
  • The index of the parent, if we haven’t found the rep. element

• We will call these UpTrees:

```
0 1 2 3
0 -1 -1 -1
-1 -1 -1 -1
```
# UpTrees

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Disjoint Sets

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</table>
Disjoint Sets Find

int DisjointSets::find() {
    if ( s[i] < 0 ) { return i; }
    else { return _find( s[i] ); }
}

Running time?

What is the ideal UpTree?
Disjoint Sets Union

```
void DisjointSets::union(int r1, int r2) {
}
```
Disjoint Sets – Union
Disjoint Sets – Smart Union

**Union by height**

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**Idea:** Keep the height of the tree as small as possible.
Disjoint Sets – Smart Union

**Union by height**

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**Union by size**

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Idea: Keep the height of the tree as small as possible.

Idea: Minimize the number of nodes that increase in height

Both guarantee the height of the tree is: ______________.
Disjoint Sets Find

```cpp
int DisjointSets::find(int i) {
    if ( s[i] < 0 ) { return i; }
    else { return _find( s[i] ); }
}
```

```cpp
void DisjointSets::unionBySize(int root1, int root2) {
    int newSize = arr_[root1] + arr_[root2];

    // If arr_[root1] is less than (more negative), it is the larger set;
    // we union the smaller set, root2, with root1.
    if ( arr_[root1] < arr_[root2] ) {
        arr_[root2] = root1;
        arr_[root1] = newSize;
    }
    // Otherwise, do the opposite:
    else {
        arr_[root1] = root2;
        arr_[root2] = newSize;
    }
}
```
Path Compression
Disjoint Sets Analysis

The **iterated log** function:

*The number of times you can take a log of a number.*

\[
\log^*(n) = \\
0 \quad , \quad n \leq 1 \\
1 + \log^*(\log(n)) \quad , \quad n > 1
\]

What is \( \lg^*(2^{65536}) \)?
Disjoint Sets Analysis

In a Disjoint Sets implemented with smart unions and path compression on find:

Any sequence of m union and find operations result in the worse case running time of O(___________), where n is the number of items in the Disjoint Sets.