Week 6 Tutorial Solutions

8.1a Paths

(Recall that a walk includes a sequence of nodes and a sequence of edges, but that usually we can give just one of those. I prefer giving the sequence of vertices, but I demonstrate the other style in the first of the bullet points below.)

- (A, F): The possible paths are 654 and 12354. Non-path walks include 11654 and 123612354. *(Recall that every path is also a walk.)*
- (F, E): The possible paths are $FDCE$ and $FDCABE$. One non-path walk is $FDCDCE$.
- (B, D): The possible paths are $BACD$ and $BECD$. One non-path walk is $BACEBECABACDFD$.
- (B, F): The possible paths are $BACDF$ and $BECDF$. One non-path walk is $BACACDF$.

8.3b Graph Connectivity

There are three connected components: the solitary node g, the solitary node h, and then everything else.

8.4 Graph Diameters

- K_n: 0 if $n = 1$, 1 otherwise. K_1 has just one vertex (which is at distance 0 from itself), so it has diameter 0. For any larger complete graph, any two distinct nodes are at distance 1 because there is an edge from every node to every other. The diameter is thus 1 (regardless of how large n is).
- C_n: $\lfloor \frac{n}{2} \rfloor$. For even n, the maximum distance is $\frac{n}{2}$, i.e. the distance between two nodes that are exactly opposite each other. For odd n, the maximum distance is still between nodes that are as close to opposite as possible, but those nodes aren't quite opposite and the two paths between them are of lengths $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. The distance between them, and hence the diameter of the graph, is the smaller of the two, $\lfloor \frac{n}{2} \rfloor$.

Thus in both even and odd cases, the diameter can be expressed as $\lfloor \frac{n}{2} \rfloor$.

- W_n: 1 if $n = 3$, 2 otherwise. For $n = 3$, W_n is just K_4 which as we saw above has diameter 1. For any larger n, there are non-adjacent nodes in the rim so the diameter must be larger than 1, but there is a short path from any node to any other that goes through the 'hub' of the wheel, so the diameter is 2.

8.5 Euler circuits

1. One possible circuit is $ablefijmcdkgh$.

2. No Euler circuit is possible because there is at least one node (S) with odd degree.

9.1b Isomorphic or not?

No isomorphism is possible: in B_1 there are two vertices of degree 3 (B and D) and they are not adjacent, while in B_2 there are also two degree-3 vertices (3 and 6) but they *are* adjacent.

(There are other features you could use to prove non-isomorphism. For example, in B_2 the three nodes of degree 4 $(1, 4, 5)$ are all adjacent to each other; B_1 also has three nodes of degree 4 (F, E, A) but there is no edge between A and F.)

9.2 Proving non-isomorphism

b) B_1 has no node with degree 3, while B_2 does (node 3).

c) These are actually the same exact graphs as in 9.1b; see that solution.

9.3a Counting isomorphisms

We can permute y, p, z (3! ways to do this), v, u (2!), and a, b, c (3!); w must map to itself. There are thus $3! \cdot 2! \cdot 3! = 72$ isomorphisms.

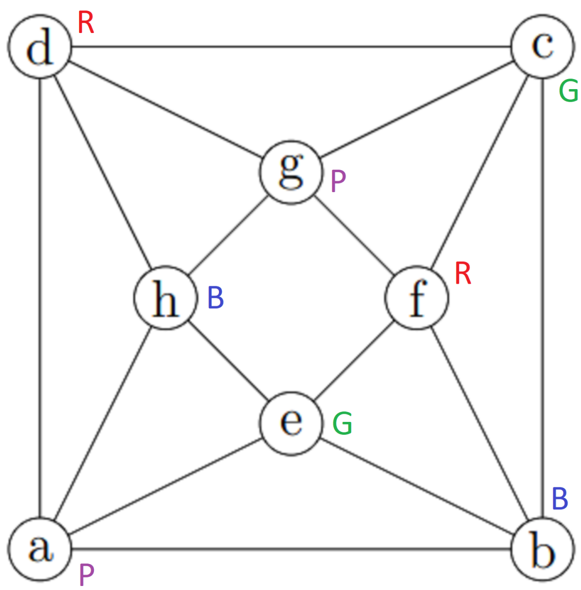

Figure 1: A four-coloring of the graph.

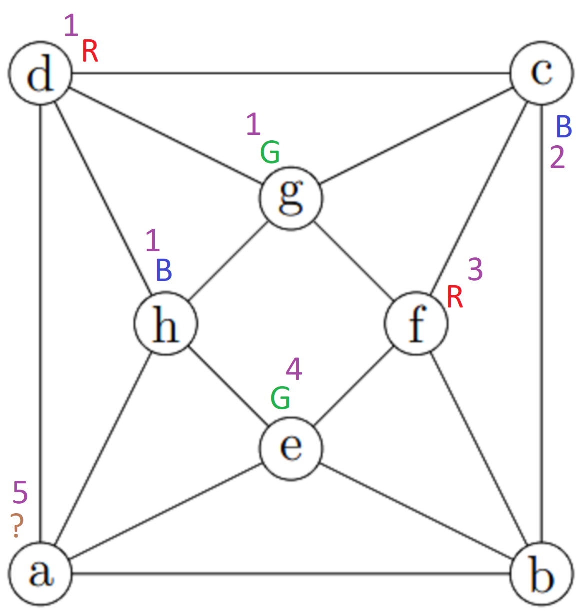

Figure 2: There is no three-coloring. (numbers correspond to steps in the proof)

10.2d Chromatic Number

Claim: the chromatic number $\chi(D)$ is 4. Proof: Figure 1 provides an upper bound of 4 by showing an explicit four-coloring, so it remains to show that the graph cannot be colored with 3 colors. We prove this as follows (see Figure 2 for a visualization): (1) Any 3-coloring must assign different colors to d, g, h; without loss of generality we call those three colors Red, Green, and Blue, respectively. (2) c is adjacent to d (Red) and g (Green), so it must be Blue. (3) f is adjacent to c (Blue) and g (Green), so it must be Red. (4) e is adjacent to f (Red) and h (Blue), so it must be Green. (5) Finally, a is adjacent to nodes of all three colors (d, h, e), so there is no possible color for a.

(Commentary: Notice that it would not *be enough to just have argued that* one particular *attempt at coloring with three colors didn't work. Instead, we argued that* every *attempt at three-coloring would run into this problem. This could also be formalized as a "proof by contradiction", but we haven't covered those yet.)*

10.1b Set Equality Proofs

We will proceed by proving that each of the two sets is a subset of the other.

Subclaim: $X \subseteq Y$. Proof: Let z be an element of X. Then by definition of X, $z = 10x + 15y$ for some integers x, y. Factoring out the 5, we get $z = 5(2x + 3y)$. $2x + 3y$ is an integer since x, y are integers, so $z \in Y$. ■

Subclaim: $Y \subseteq X$. Proof: Let w be an element of Y. Then $w = 5k$ for some integer k. Then by algebra, $w = 10(-k) + 15k$. Since k is an integer, $-k$ is also an integer, so we see $w \in X$. ■

Since each set is a subset of the other, the two sets are equal, QED.

Additional problem: bounding vertices in 7-edge connected graph

We first prove the following lemma: A graph with n vertices has at most $n(n-1)/2$ edges. Proof: Each edge can be uniquely described by choosing one of the n possible endpoints, then one of the $n-1$ possible second endpoints, and then dividing by 2 because the order of the endpoints doesn't matter.

Claim: The best lower bound is 5 vertices. Proof: The lower bound cannot be greater than 5 since there exist 5-vertex graphs with 7 edges (e.g. delete any 3 edges from K_5). And by the lemma above, any graph with 4 or fewer vertices has at most 6 edges, so G has more than 4 vertices. 5 is thus the tightest possible lower bound.

Claim: The best upper bound is 8 vertices. Justification *(proving this properly uses induction, which we haven't covered yet)*: The upper bound cannot be smaller than 8 because there exist 8-vertex graphs with 7 edges (e.g. the line graph L_8: $\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet$). And G can't have more than 8 vertices because then it wouldn't be connected. Informally, each new edge in a graph reduces the number of connected components by at most 1, so if we start with $n > 8$ vertices then after adding 7 edges we will have at least $n - 7 > 1$ connected components, so the graph will still not be connected. 8 is thus the tightest possible upper bound.