

## Week 5 Tutorial Solutions

### 5.1 Nested Quantifiers

- b) This is true: 0 is a value for  $x$  such that, for every possible value of  $y$ ,  $xy = x$ . (*The domain is very important here - if instead it were e.g. positive integers, then the statement would be false.*)
- d) This is false: no matter what value is chosen for  $y$ ,  $x = y + 1$  is a value of  $x$  which differs from  $y$  by more than 0.01. (*This argument is showing that the negation of the given statement is true:  $\forall y \in \mathbb{Q}, \exists x \in \mathbb{R}, |x - y| > 0.01$ .*)

### 5.2 Meditating on the Definition of Onto

- a) This translates to “for every element in the co-domain, there is an element of the domain which maps to it”, which is just the definition of *onto*.
- b) This translates to “every element in the domain maps to (at least) one element of the co-domain”. This statement is true of *all* functions, so it doesn’t represent any interesting named concept.
- c) This translates to “there is an element of the co-domain which is mapped to from every element of the domain”. In other words, this represents the concept of a function being *constant* - e.g. this statement is true of the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  defined by  $f(x) = 42$ .
- d) This translates to “there is an element of the domain which maps to *every* element of the co-domain”. Since every input to a function must map to *exactly* one output, the only way this statement can ever be true is if the co-domain has only one element. (*In pure math I can’t think of any reason to define a function with a co-domain containing only one element, but some programming languages like Haskell do actually find it useful to do this.*)

### 5.4 Abstract onto proof

We need to show that  $g$  is onto, i.e. that for every  $n \in \mathbb{N}$ , there is some pair  $(x, y) \in \mathbb{N}^2$  such that  $g(x, y) = n$ . So let  $n$  be an arbitrary natural number. Then since  $f$  is onto, there is some  $k \in \mathbb{N}$  such that  $f(k) = n$ . Similarly, since  $f$  is onto, there is some  $z \in \mathbb{N}$  such that  $f(z) = 1$ . So then  $g(k, z) = f(k)f(z) = n \cdot 1 = n$ , which is what we needed to show.

### 7.3 Abstract proof using Composition

- a) Suppose that  $f \circ g$  is onto and  $f$  is one-to-one. Let  $y$  be an element of  $B$ , and consider  $z = f(y)$ . Since  $f \circ g$  is onto, there is an element  $k \in A$  such that  $(f \circ g)(k) = z$ , i.e.  $f(g(k)) = z$ . Since  $f$  is one-to-one and  $f(g(k)) = z = f(y)$ , it follows that  $g(k) = y$ . Thus we have found an element of  $A$  which maps to  $y$ , so  $g$  is onto. QED.
- b) Let  $A = \{a\}$ ,  $B = \{b, z\}$ ,  $C = \{c\}$ . Let  $f$  and  $g$  be the constant functions  $f(x) = c$  and  $g(x) = b$ . Then  $f \circ g$  is onto (because  $f(g(a)) = c$ ), but  $g$  is not onto (because there is no input which gives the output  $z$ ).

### 7.5c Pigeonhole Principle Proof

Split the 10 nights into 5 pairs of consecutive nights (nights 1 and 2, nights 3 and 4, etc). Each of the 61 hours occurs during one of these five pairs of nights. Since there are more than 12 times as many hours as pairs, there must be at least one pair which ends up with 13 or more hours, which was what we needed to prove.

**Warning:** *There is a common incorrect solution which begins by using the pigeonhole principle to show that some night has 7 or more hours. While this step isn't wrong, I do not know of any correct solution that continues from there. If your proof uses this step, check whether your proof could be adapted to prove a similar claim where the number of nights is reduced from 10 to 3 and the number of hours from 61 to 19. If so, your proof must have a flaw somewhere, because that version of the claim is false!*

### Additional problem

*(The intuition behind both parts of this problem is that if we “reverse the arrows” of  $f$ , we get something that isn't quite a valid function  $g$  but is pretty close. We just have to figure out what else to fix each time. See Figures 1 and 2 for visualizations, but remember these are specific examples, not proofs.)*

- a) Since  $f$  is onto, for every  $s \in B$  there is at least one  $t \in A$  such that  $f(t) = s$ . We can thus define  $g(s)$  to be the *minimum* value  $t \in A$  such that  $f(t) = s$ . (Note that if  $f$  weren't onto then this wouldn't be a valid function definition at all - what is the minimum of the empty set?) Now we will prove that  $g$  is one-to-one. Let  $x, y$  be elements of  $B$ , and let

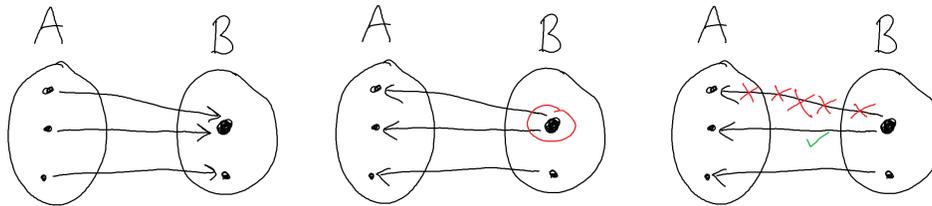


Figure 1: Visualization for part (a). Left: an onto function  $f$ . Middle: reversing the arrows is not a function because an input has multiple outputs. Right: arbitrarily keeping only one output for each input (the smallest) makes  $g$  a function, and it is one-to-one.

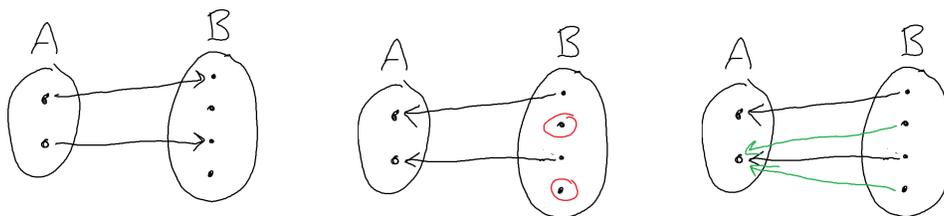


Figure 2: Visualization for part (b). Left: a one-to-one function  $f$ . Middle: reversing the arrows gives us an onto function from  $f(A)$  to  $A$ , but it is not a function at all from  $B$  to  $A$  because some inputs in  $B$  would have no output. Right: arbitrarily assigning an output (the smallest element of  $A$ ) to all the missing inputs makes  $g$  a function, and it is onto.

$g(x) = g(y) = z$  for some  $z \in A$ . Then by the definition of  $g$ ,  $f(z) = x$  and also  $f(z) = y$ , so  $x = y$ , QED.

- b) Define  $f^{-1} : f(A) \rightarrow A$  as the inverse of  $f$  where such an inverse is defined - that is, the function which maps each element of  $f(A)$  to its pre-image in  $A$ . (See middle of Figure 2 for an example of what  $f^{-1}$  might look like.)  $f^{-1}$  is a well-defined function because  $f$  is one-to-one, so each element of the image  $f(A)$  has a *unique* pre-image. Then we can construct  $g$  as follows:

$$g(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f(A) \\ \text{min}(A) & \text{otherwise} \end{cases}$$

Now we will prove that  $g$  is onto. Let  $y$  be an element of  $A$ , and then we need to find an  $x$  such that  $g(x) = y$ . Consider  $z = f(y)$ . By definition  $z$  is in the image  $f(A)$ , so  $f^{-1}(z) = y$ . Then by the definition of  $g$ ,  $g(z) = f^{-1}(z) = y$ , so  $z$  is the element  $x$  we needed, QED.