

Week 4 Tutorial Problems

1. Constructing a concrete relation

Construct a relation R on the set $\{1, 2, 3\}$ such that all the following are true:

- $1R2$
- R is symmetric
- R is transitive
- R is not an equivalence relation

(You can specify the relation however you want: a diagram with arrows, a table of related pairs, etc.)

2. Discussion manual problems

Do the following problems from the discussion manual:

- 4.2 parts (a) and (b)
- 4.3 part (a), except you do not need to prove the relation is an equivalence relation.
- 4.3 part (b)

3. Abstract relation proof

Let R and S be symmetric relations on some set A . Define a relation \sim on A such that $x \sim y$ if and only if xRy and $\neg(xSy)$. Prove that \sim is symmetric.

Solutions

1. Constructing a concrete relation

R is the relation which relates a to b for each row in this table:

a	b
1	2
2	1
1	1
2	2

(Commentary: It turns out the above solution is the only relation that satisfies the given conditions. To construct it, we can apply the conditions one at a time as follows: First, we know that $1R2$. Then to make R symmetric, we have to also include $2R1$. Then to make R transitive, we have to include $1R1$ and $2R2$. (Remember that the definition of transitive does not require that x, y, z be distinct.) Finally we check over all the conditions again to confirm we are done. You did not have to demonstrate that this is the only possible R , but notice that if you include $3R3$ then the relation would be reflexive and thus an equivalence relation, and if you make any other additional pair of elements related, then either you will not add enough to restore symmetry and transitivity, or you will discover that everything has to be related to everything else, which is again an equivalence relation.)

2. Discussion manual problems

- 4.2a) • R is a partial order. It is reflexive because every letter has a self-loop, it is antisymmetric because no arrow has a matching arrow in the reverse direction, and we can check transitivity by confirming that for every two arrows that can be 'chained' one after another (like ARC and CRB), the corresponding third arrow (ARB) is present.

(Commentary: To check that a relation defined by an arbitrary diagram is transitive, it helps to notice that the presence of a self-loop will never cause the relation to be non-transitive, because the following two statements are always true: $xRx \wedge xRz \rightarrow xRz$, and $xRy \wedge yRy \rightarrow xRy$. Note that the absence of a self-loop can sometimes make a relation non-transitive, as discussed in the lecture.)

- T is not a partial order because it is not transitive: we have CTB and BTB , yet we do not have CTB .

4.2b) We need to show that \preceq is reflexive, antisymmetric, and transitive.

- Claim: \preceq is reflexive. Consider an element $(a, b) \in \mathbb{Z}^2$. We know that $a \leq a$ and $b \leq b$, so by the definition of \preceq , $(a, b) \preceq (a, b)$.
- Claim: \preceq is antisymmetric. Consider elements $(a, b), (c, d) \in \mathbb{Z}^2$ such that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$. Then by the definition of \preceq , we have $a \leq c$, $b \leq d$, $c \leq a$, and $d \leq b$. Since $a \leq c$ and

$c \leq a$, we know $a = c$. By similar logic, we know $b = d$. Thus $(a, b) = (c, d)$.

- Claim: \preceq is transitive. Consider elements $(a, b), (c, d), (e, f) \in \mathbb{Z}^2$ such that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$. Then by the definition of \preceq , we have $a \leq c, b \leq d, c \leq e$, and $d \leq f$. Then by transitivity of \leq we get $a \leq e$ and $b \leq f$, so $(a, b) \preceq (e, f)$.

- 4.3a) By the definition of \sim , $[(1, 3)]$ contains every (c, d) pair where $1 + d = 3 + c$, i.e. where $d - c = 2$. Similarly, we see that $[(0, 4)]$ contains all pairs where $d - c = 4$, and $[(2, 4)]$ contains all pairs where $d - c = 2$ (i.e. it is the same as our earlier $[(1, 3)]$). More generally, there is one equivalence class $[(0, k)] = \{(c, d) | d - c = k\}$ for each integer k .

(Commentary: Each equivalence class is an lower-left-to-upper-right diagonal line in the plane. I chose $[(0, k)]$ as my ‘canonical’ name for each of these diagonal lines (i.e. naming the line after the place where it crosses the y-axis), but we could just as well have taken our pick of infinitely many other possible names, including e.g. $[(-k, 0)]$, naming after the place where it crosses the x-axis).

- 4.3b) By the definition of \sim , $[2]$ contains every integer y where $4 | (3 \cdot 2 + 5y)$. Since $6 + 5y$ and $y - 2$ differ by a multiple of 4 (i.e. $4(y - 2)$), we see that $4 | (6 + 5y)$ iff $4 | (y - 2)$, so $[2]$ contains all integers that are 2 more than a multiple of 4, i.e. $[2]$ is precisely the congruence class of 2 mod 4.

Similarly, $[3]$ contains every integer y where $4 | (3 \cdot 3 + 5y)$. Since $9 + 5y$ and $y - 3$ differ by a multiple of 4 (i.e. $4(y - 3)$), we see that $4 | (9 + 5y)$ iff $4 | (y - 3)$, so $[3]$ contains all integers that are 3 more than a multiple of 4, i.e. $[3]$ is precisely the congruence class of 3 mod 4.

By continuing the same logic, we see that there are only 4 equivalence classes and these are the congruence classes of 0, 1, 2, and 3 mod 4. In fact \sim is just equivalence mod 4, i.e. $x \sim y$ if and only if $x \equiv y \pmod{4}$.

We will now show \sim is an equivalence relation by showing it is reflexive, symmetric, and transitive:

- Claim: \sim is reflexive. Consider an integer x . $3x + 5x = 8x = 4(2x)$, so $4 | 3x + 5x$ and thus $x \sim x$.

- Claim: \sim is symmetric. Consider integers x, y such that $x \neq y$ and $x \sim y$. Then by definition of \sim , $4|3x+5y$, i.e. $3x+5y = 4k$ for some integer k . Then $9x+15y = 12k$, so $5x+3y = 12k-4x-12y = 4(3k-x-3y)$. Thus $3y+5x$ is a multiple of 4, so $y \sim x$.

(Commentary: This is probably the hardest of the properties to prove. You start with $3x+5y = 4k$ and you know you have to end with $5x+3y = 4(\text{something})$; it may take significant experimenting with adding equations, multiplying by constants, etc to eventually find the solution here. Don't worry about getting everything to fit together in logical order until you've first found all the pieces in your scratchwork.)

- Claim: \sim is transitive. Consider integers x, y, z such that $x \sim y$ and $y \sim z$. Then by definition of \sim , $4|3x+5y$, and $4|3y+5z$. i.e. $3x+5y = 4k$ and $3y+5z = 4m$ for some integers k, m . Adding the equations gives us $3x+8y+5z = 4k+4m$, so $3x+5z = 4(k+m-2y)$ and thus $x \sim z$.

3. Abstract relation proof

Let a, b be distinct elements of A such that $a \sim b$. By definition of \sim , aRb and $\neg(aSb)$. Since aRb and R is symmetric, we have bRa . Since $\neg(aSb)$ and S is symmetric, we have $\neg(bSa)$. *(This comes from using the definition of symmetry in its contrapositive form, i.e. for all $x, y \in A$ such that $x \neq y$, if $\neg(yRx)$, then $\neg(xRy)$.)* Finally, since we have bRa and $\neg(bSa)$, by the definition of \sim , $b \sim a$.