CS 173 Lecture 6b: Special Kinds of Relations + Proofs

$R \subseteq A \times A$

Definition: A relation R on A is <u>partial order</u> if R is (i) reflexive, (ii) antisymmetric, (iii) transitive

$\forall a \in A,\ aRa$ — $(a,a) \in R$

$\forall a,b \in A,\ aRb \land bRa \to a=b$

$\forall a,b,c \in A,\ aRb \land bRc \to aRc$

Consider divisibility on $\mathbb{Z}_+ = \{1,2,3,\dots\}$

Claim: divisibility on $\mathbb{Z}_+$ is a partial order.

Proof: (i) Let a be a positive integer.
Since $a = a \cdot 1$, $a \mid a$.
So divisibility on $\mathbb{Z}_+$ is reflexive.

(ii) Let a, b be positive integers such that $a \mid b$, $b \mid a$. We aim to show $a=b$.
By definition of divisibility, there exist integers k & l such that $b = ak$, and $a = bl$. Then $a = akl$. So $kl = 1$.
Either $k = l = -1$ or $k = l = 1$. But since $b > 0$, $a > 0$, $b = ak$, $k = 1$. So $b = a$.
So divisibility on $\mathbb{Z}_+$ is antisymmetric.

(iii) Let a, b, c be positive integers such that $a \mid b$ and $b \mid c$. By Theorem from lecture 3a, $a \mid c$. So divisibility on $\mathbb{Z}_+$ is transitive.

We have shown that this relation is a partial order. $\square$

A relation R on A is a <u>linear order</u> if it is a partial order <u>and</u> every pair $a, b \in A$ is <u>comparable</u>, i.e., either aRb or bRa

$(a,b) \in R$ $\qquad$ $(b,a) \in R$

Claim: divisibility on $\mathbb{Z}_+$ is not a linear order.

Proof: We will show that there exist $a, b \in \mathbb{Z}_+$ such that a & b are not comparable.

Let $a = 4$, $b = 6$. Then $4 \nmid 6$ and $6 \nmid 4$

such that a & b are not comparable.

Let $a=4$, $b=6$. Then $4\nmid 6$ and $6\nmid 4$.

So divisibility on $\mathbb{Z}_+$ is not a linear order. $\square$

Claim: $\leq$ on $\mathbb{R}$ is a linear order.

Proof? (Verification) (i) reflexivity: $a\leq a$ for all $a\in\mathbb{R}$

(ii) antisymmetry: $a\leq b$ and $b\leq a$, then $a=b$ for all $a,b\in\mathbb{R}$.

(iii) transitivity: $a\leq b$, and $b\leq c$, then $a\leq c$. for all $a,b,c\in\mathbb{R}$.

(iv) comparability: For all $a,b\in\mathbb{R}$, $a\leq b$ or $b\leq a$. $\square$.

R on A is a strict partial order if

(i) irreflexive, (ii) antisymmetric, (iii) transitive.

$\forall a,\ a\not R a$

A = courses at UIUC $\quad R=\{(a,b)\in A\times A:\ a \text{ is a prereq for } b\}$

Claim: R is a strict partial order.

Proof: (i) No course can be its own prereq, so for all courses a, $a\not R a$.

(ii) Suppose we have two courses a & b such that a is a prereq for b and b is a prereq for a. This is never true, so the statement $\forall a,b\in A\ \underbrace{aRb \,\&\, bRa}_{\text{False.}}\rightarrow a=b$ is vacuously true.

(iii) If a is a prereq for b, and b is a prereq for c, one needs to take a before taking c.

So prereq relation is a strict partial order. $\square$

A relation R on A is an equivalence relation if it is

(i) reflexive (ii) symmetry (iii) transitivity.

$\forall a,b\in A$
$aRb\rightarrow bRa$

$aRb \Rightarrow bRa$

divisibility on $\mathbb{Z}_+$ is NOT an eq. relation because it's not symmetric: $2 \mid 4$ but $4 \nmid 2$.

i.e. $\exists\, a, b \in A$ s.t. aRb & $b \not R a$

Fix $k \in \mathbb{Z}$. Then congruence mod k is an equivalence relation.

Proof: (i) Reflexive: Let a be an integer.

$a \equiv a \mod k$ since $k \mid 0$

$a-a$

$0 \cdot k = 0$ $\forall k \in \mathbb{Z}$

(ii) Symmetry: Let a, b be integers such that $a \equiv b \mod k$.

So $k \mid (b-a)$, i.e. there exists an integer ℓ s.t. $b-a = k\ell$. So $a-b = k(-\ell)$, so $k \mid a-b$, i.e., $b \equiv a \mod k$.

(iii) Transitivity: Let a, b, c be integers s.t. $a \equiv b \mod k$, $b \equiv c \mod k$.

Since $a \equiv b \mod k$, $k \mid b-a$ so there exists $\ell \in \mathbb{Z}$ s.t. $\underline{b-a = k\ell}$, and since $b \equiv c \mod k$, there exists $m \in \mathbb{Z}$ s.t. $\underline{c-b = km}$. So

$$\begin{aligned} c-a &= (c-b)+(b-a) \\ &= km + k\ell \\ &= k(m+\ell). \end{aligned}$$

Thus $k \mid c-a$, i.e. $a \equiv c \mod k$.

We have shown that $\equiv \mod k$ is an eq. rel. $\square$.

Recall equivalence classes of integers mod k.

$$[x] = \{ y \in \mathbb{Z} : x \equiv y \mod k \}.$$

Given any equivalence relation R on A, the equivalence classes of R are:

equivalence classes of R are:

$$[x] = \{ y \in A : xRy \}.$$

(i) $[x] = [y]$ iff xRy.

(ii) if $[x] = [y]$, and $[y] = [z]$, then $[x] = [z]$.