CS 173 Episode IV: A New Proof Technique

Part A: In(tro)duction

Principle of Induction

Let P be a predicate defined over $\mathbb{Z}$.

Suppose that $P(a)$ is true, for some $a \in \mathbb{Z}$.

and $P(a) \wedge P(a+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ $\forall n \in \mathbb{Z}$, $n \geq a$.

Then $P(n)$ is true for all $n \in \mathbb{Z}$, $n \geq a$.

Proof by induction:

Suppose we want to prove $P(n)$ for all $n \geq a$.

Proof. By induction.

Base Case: Prove $P(a)$ is true.

Inductive Step.

Assume $P(a) \wedge P(a+1) \wedge \cdots \wedge P(n-1)$.

Inductive Hypothesis

$\vdots$

Therefore $P(n)$.

By induction, $P(n)$ holds for $n \geq a$.

$P(n)$

Claim: For integers $n \geq 2$, $\boxed{\sum_{i=2}^{n} i 2^i = (n-1) 2^{n+1}}$.

Proof: Case $n=2$: $\sum_{i=2}^{2} i2^i = 2 \cdot 2^2 = 8$.

$(2-1)2^{2+1} = 1 \cdot 2^3 = 8$.

So $\sum_{i=2}^{2} i2^i = (2-1)2^{2+1}$.

Inductive Step: Case $n > 2$.

Assume that for $k \in \{2, 3, \cdots n-1\}$ that $\sum_{i=2}^{k} i2^i = (k-1)2^{k+1}$.] I.H.

that $\sum_{i=2}^{k} i2^i = (k-1)2^{k+1}$.

$$\sum_{i=2}^{n} i2^i = n2^n + (n-1)2^{n-1} + (n-2)2^{n-2} + \dots + 2\cdot 2^2$$
$$= n2^n + \sum_{i=2}^{n-1} i2^i$$

P(n-1): $\sum_{i=2}^{n-1} i2^i = ((n-1)-1)2^{(n-1)+1}$

$$= n2^n + (n-2)2^n$$
$$= (2n-2)2^n$$
$$= (n-1)\cdot 2\cdot 2^n$$
$$= (n-1)2^{n+1}.$$

Therefore $\sum_{i=2}^{n} i2^i = (n-1)2^{n+1}$

By induction, $\sum_{i=2}^{n} i2^i = (n-1)2^{n+1}$ for all integers $n \geq 2$. $\square$

Claim: For all integers $n \geq 18$, there exist non-negative integers k & l such that $n = 3k + 10l$.

Proof. Case $n=18$. Then $18 = 3\cdot 6$, so for $k=6$ & $l=0$, $18 = 3k + 10l$.

Case $n=19$. Then $19 = 3\cdot 3 + 10$, so for $k=3$ & $l=1$, $19 = 3k+10l$

Case $n=20$. Then $20 = 2\cdot 10$, so for $k=0$ & $l=2$, $20 = 3k+10l$.

Inductive Step: Assume that for $m \in \{18, 19, \dots, n-1\}$ that there exist k & l such that $m = 3k + 10l$

Case $n \geq 21$.

$n = n-3+3$. Since $n \geq 21$, the inductive hypothesis implies that $n-3 = 3k + 10l$ for some non-negative integers k & l.

So $n = 3(k+1) + 10l$.

So by induction, for all integers $n \geq 18$, there exist non-neg ints k & l such that $n = 3k + 10l$. $\square$