Homework 6
Discrete Structures
CS 173 [B] : Fall 2015
Released: Fri Apr 10
Due: Fri Apr 17, 5:00 PM

Submit on Moodle.

PART 1 (Machine-Graded Problems) on Moodle. [25 points]
PART 2 [75 points]

1. Recurrence Relation [20 points]

Recall that \( \binom{n}{k} \) is the number of subsets of size \( k \) that a set of size \( n \) has.

(a) Use mathematical induction to prove that, for all \( n, k \in \mathbb{N} \) such that \( k \leq n \), we have \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \), based on the following: \( \forall n \in \mathbb{N}, \binom{n}{0} = \binom{n}{n} = 1 \); and, for \( n \geq 1 \), \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \) (which we obtained by considering separately the subsets of size \( k \) that contain and do not contain a fixed element from the set).

(b) Above, \( \binom{n}{k} \) was expressed in terms of \( \binom{n-1}{i} \) for two different values of \( i \). Use a similar argument to express \( \binom{n}{k} \) in terms of \( \binom{n-2}{i} \) for different values of \( i \) (for \( n \geq 2 \)).

[Hint: Alternately, note that \( \binom{n}{k} \) is the coefficient of \( x^k \) in the expansion of \( (1 + x)^n = (1 + x)^2 \cdot (1 + x)^{n-2} \).]

2. Partitions from Onto Functions. [20 points]

Consider the following definitions.

- For a function \( f : A \to B \), let \( \hat{f} : A \to \text{Image}(f) \) be the unique onto function such that \( \forall x \in A \ f(x) = \hat{f}(x) \).
- For a function \( g : A \to C \), let the pre-image function \( PI_g : C \to \mathcal{P}(A) \) be defined by \( PI_g(y) = \{ x \mid f(x) = y \} \).
- For a function \( f : A \to B \), let the “pre-image partition” of \( A \), be defined as \( PP_f = \text{Image}(PI_f) \).
- Define an equivalence relation \( \sim \) between functions \( f_1 : A \to B \) and \( f_2 : A \to B \) as follows: \( f_1 \sim f_2 \) if \( PP_{f_1} = PP_{f_2} \).

Answer the following with respect to the above definitions.

(a) Suppose \( A = \{a, b, c\} \) and \( B = \{1, 2, 3\} \). Consider \( f : A \to B \) defined as \( f(a) = f(b) = 1 \) and \( f(c) = 2 \). Also, let \( f' : A \to B \) be defined as \( f'(a) = f'(b) = 3 \) and \( f'(c) = 2 \)

i. Describe the functions \( f \) and \( f' \).
ii. Describe the functions \( PI_f \) and \( PI_{f'} \).
iii. Describe the partitions $PP_f$ and $PP_{f'}$.

(b) Let $f : A \to B$, where $|A| = n$, $|B| = k$ and $|\text{Image}(f)| = i$. Then how many functions $f'$ are there such that $f \sim f'$? Justify your answer.

3. Lottery

Counting is intimately connected to computing the probability of various events. In this problem we shall use counting to calculate the probability of winning lotteries.

In a certain kind of lottery, each player submits a sequence of $n$ digits (between 0 and 9). A player wins a grand prize if her submission exactly matches a sequence of $n$ digits selected by a random mechanical process. She wins a smaller prize if only $n - 1$ digits are matched (e.g., for $n = 4$, if the submission is 1248 but the machine chooses 1298, then a small prize is awarded).

(a) How many ways can the mechanical process choose a sequence of $n$ digits? Use this to compute the probability of a player (who has submitted a single sequence) winning the large prize, assuming that the mechanical process chooses each possible sequence equally likely (i.e., uniformly at random).

[Hint: You can use the following fact regarding probability. If one item is chosen out of $N$ possible items uniformly at random, then the probability of it being any priori fixed item is $1/N$.]

(b) For any sequence of $n$ digits that a player picks, how many sequences are there which, if chosen by the mechanical process, would result in the player winning a small prize? Use this to compute the probability that a player (who has submitted a single sequence) wins the small prize.

[Hint: The probability in this case is $p/N$, where $p$ is the number of sequences, which if chosen by the mechanical process, leads to a small prize, and $N$ is the total number of all possible sequences that the mechanical process can choose.]

4. Sorted Strings

Consider strings made up of lowercase letters, a-z. We say that a string is a “sorted string” if the letters in it appear in alphabetic order. For instance, bbn and tux are sorted strings, but ibm is not.

(a) How many sorted strings of length 3 are there?

[Hint: Can you relate a sorted string to a multi-set?]

(b) How many sorted strings of length 3 are there in which no letter repeats? (Thus bbn should not be counted, but tux should be.)