Homework 6
Discrete Structures
CS 173 [B] : Fall 2015
Released: Fri Apr 10
Due: Fri Apr 17, 5:00 PM

Submit on Moodle.

PART 1 (Machine-Graded Problems) on Moodle. [25 points]

PART 2 [75 points]

1. Recurrence Relation [20 points]

Recall that \((n \choose k)\) is the number of subsets of size \(k\) that a set of size \(n\) has.

(a) Use mathematical induction to prove that, for all \(n, k \in \mathbb{N}\) such that \(k \leq n\), we have \((n \choose k) = \frac{n!}{k!(n-k)!}\), based on the following: \(\forall n \in \mathbb{N}, (n \choose 0) = (n \choose n) = 1\); and, for \(n \geq 1\), \((n \choose k) = (n-1 \choose k-1) + (n-1 \choose k)\) (which we obtained by considering separately the subsets of size \(k\) that contain and do not contain a fixed element from the set).

Solution: We need to prove that \(\forall n \in \mathbb{N}\) the following holds: \(\forall k \in \mathbb{N}, k \leq n, (n \choose k) = \frac{n!}{k!(n-k)!}\).

We prove this statement by induction on \(n\).

The base case: Consider \(n = 0\). Then, the only value of \(k \in \mathbb{N}, k \leq n\) is \(k = 0\). For \(n = 0, k = 0\), we have \((0 \choose 0) = 1\) (by a base case of the recursive definition) and \(\frac{0!}{0!(0-0)!} = 1\).

Induction step:
Suppose that for some \(n_0 \in \mathbb{N}\), for all \(n \leq n_0\), it holds that \(\forall k \in \mathbb{N}, k \leq n, (n \choose k) = \frac{n!}{k!(n-k)!}\).

Then we shall prove that, for all \(k \in \{0, \ldots, n_0 + 1\}, (n_0 + 1 \choose k) = \frac{(n_0 + 1)!}{k!(n_0 + 1 - k)!}\).

First, for \(k = 0\) and \(k = n_0 + 1\) this is given by the base cases of the recurrence relation.

Now consider \(k \in \{1, \ldots, n_0\}. Since n_0 + 1 \geq 1, we can apply the recurrence relation to obtain

\[
(n_0 + 1 \choose k) = (n_0 \choose k - 1) + (n_0 \choose k)
\]

\[
= \frac{n_0!}{(k-1)!(n_0 - k + 1)!} + \frac{n_0!}{k!(n_0 - k)!} \quad \text{by IH, since } 0 \leq k - 1, k \leq n_0
\]

\[
= \frac{n_0!}{(k-1)!(n_0 - k)!} \left(\frac{1}{n_0 - k + 1} + \frac{1}{k}\right)
\]

\[
= \frac{n_0!}{(k-1)!(n_0 - k)!} \left(\frac{n_0 + 1}{n_0 - k + 1 \cdot k}\right)
\]

\[
= \frac{(k \cdot (k-1)! \cdot ((n_0 - k + 1) \cdot (n_0 - k))}{(n_0 + 1)! \cdot (n_0 - k + 1)!}
\]

\[
= \frac{k \cdot (n_0 + 1)!}{(n_0 + 1)! \cdot (n_0 - k + 1)!} = \frac{n_0!}{k!(n_0 - k)!}
\]
Note that the induction hypothesis could be applied to rewrite both \( \binom{n_0}{k-1} \) and \( \binom{n_0}{k} \), since \( 0 \leq k - 1 \leq n_0 \) and \( 0 \leq k \leq n_0 \) (because \( k \in \{1, \ldots, n_0\} \)).

Thus, we have shown that if the induction hypothesis holds, then for all \( k \in \{0, \ldots, n_0 + 1\} \),
\[
\binom{n_0+1}{k} = \frac{(n_0+1)!}{k!(n_0+1-k)!}.
\]

By mathematical induction, this proves that for all \( n \in \mathbb{N} \), for all \( k \in \{0, \ldots, n\} \), \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

(b) Above, \( \binom{n}{k} \) was expressed in terms of \( \binom{n-1}{i} \) for two different values of \( i \). Use a similar argument to express \( \binom{n}{k} \) in terms of \( \binom{n-2}{i} \) for different values of \( i \) (for \( n \geq 2 \)).

[Hint: Alternately, note that \( \binom{n}{k} \) is the coefficient of \( x^k \) in the expansion of \( (1 + x)^n = (1 + x)^2 \cdot (1 + x)^{n-2} \).]

**Solution:** To choose a subset of \( k \) elements from a set \( S \) with two elements \( a, b \) and \( |S| = n \), one has four options:

- Include both \( a \) and \( b \) in the subset: the remaining elements can be chosen in \( \binom{n-2}{k-2} \) ways.
- Include \( a \) but not \( b \) in the subset: the remaining elements can be chosen in \( \binom{n-2}{k-1} \) ways.
- Include \( b \) but not \( a \) in the subset: the remaining elements can be chosen in \( \binom{n-2}{k-1} \) ways.
- Include neither \( a \) nor \( b \) in the subset: the remaining elements can be chosen in \( \binom{n-2}{k} \) ways.

Summing up, there are \( \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} \) ways of choosing a subset of size \( k \) from a set of size \( n \) (with \( n \geq 2 \)).

2. **Partitions from Onto Functions.**

Consider the following definitions.

- For a function \( f : A \to B \), let \( \hat{f} : A \to \text{Image}(f) \) be the unique onto function such that \( \forall x \in A \ f(x) = \hat{f}(x) \).
- For a function \( g : A \to C \), let the pre-image function \( PI_g : C \to \mathcal{P}(A) \) be defined by \( PI_g(y) = \{ x \mid f(x) = y \} \).
- For a function \( f : A \to B \), let the “pre-image partition” of \( A \), be defined as \( PP_f = \text{Image}(PI_f) \).
- Define an equivalence relation \( \sim \) between functions \( f_1 : A \to B \) and \( f_2 : A \to B \) as follows: \( f_1 \sim f_2 \) if \( PP_{f_1} = PP_{f_2} \).

Answer the following with respect to the above definitions.

(a) Suppose \( A = \{a, b, c\} \) and \( B = \{1, 2, 3\} \). Consider \( f : A \to B \) defined as \( f(a) = f(b) = 1 \) and \( f(c) = 2 \). Also, let \( f' : A \to B \) be defined as \( f'(a) = f'(b) = 3 \) and \( f'(c) = 2 \)

i. Describe the functions \( \hat{f} \) and \( \hat{f}' \).

**Solution:** \( \hat{f} : A \to \{1, 2\} \) is defined as \( \hat{f}(a) = \hat{f}(b) = 1 \) and \( \hat{f}(c) = 2 \). \( \hat{f}' : A \to \{2, 3\} \) is defined as \( \hat{f}'(a) = \hat{f}'(b) = 3 \) and \( \hat{f}'(c) = 2 \).

ii. Describe the functions \( PI_f \) and \( PI_{f'} \).

**Solution:** \( PI_f : \{1, 2\} \to \mathcal{P}(A) \) is defined as \( PI_f(1) = \{a, b\} \), \( PI_f(2) = \{c\} \). \( PI_{f'} : \{2, 3\} \to \mathcal{P}(A) \) is defined as \( PI_{f'}(2) = \{c\} \), \( PI_{f'}(3) = \{a, b\} \).

iii. Describe the partitions \( PP_f \) and \( PP_{f'} \).

**Solution:** \( PP_f = PP_{f'} = \{\{a, b\}, \{c\}\} \).
(b) Let \( f : A \rightarrow B \), where \( |A| = n \), \( |B| = k \) and \( |\text{Image}(f)| = i \). Then how many functions \( f' \) are there such that \( f \sim f' \)? Justify your answer.

**Solution:** \( PP_f \) consists of \( i \) non-empty sets, each of which has all its members mapped to the same value in \( B \). That is, \( f \) labels each of the \( i \) sets in \( PP_f \) with a distinct element in \( B \). A function \( f' \) such that \( PP_f = PP_{f'} \) can be chosen by choosing any \( i \) distinct values as labels for the \( i \) sets in \( PP_f \). Since \( B \) has \( k \) elements, this can be done in \( P(k, i) \) ways.

Thus there are \( P(k, i) = \frac{k!}{(k-i)!} \) functions \( f' \) such that \( f \sim f' \).

3. **Lottery**

Counting is intimately connected to computing the probability of various events. In this problem we shall use counting to calculate the probability of winning lotteries.

In a certain kind of lottery, each player submits a sequence of \( n \) digits (between 0 and 9). A player wins a grand prize if her submission exactly matches a sequence of \( n \) digits selected by a random mechanical process. She wins a smaller prize if only \( n - 1 \) digits are matched (e.g., for \( n = 4 \), if the submission is 1248 but the machine chooses 1298, then a small prize is awarded).

(a) How many ways can the mechanical process choose a sequence of \( n \) digits? Use this to compute the probability of a player (who has submitted a single sequence) winning the large prize, assuming that the mechanical process chooses each possible sequence equally likely (i.e., uniformly at random).

[Hint: You can use the following fact regarding probability. If one item is chosen out of \( N \) possible items uniformly at random, then the probability of it being any priori fixed item is \( 1/N \).]

**Solution:** A sequence of \( n \) digits can be chosen in \( N = 10^n \) ways. Hence, the probability that this matches the one submitted by a given player is \( 1/10^n \).

(b) For any sequence of \( n \) digits that a player picks, how many sequences are there which, if chosen by the mechanical process, would result in the player winning a small prize? Use this to compute the probability that a player (who has submitted a single sequence) wins the small prize.

[Hint: The probability in this case is \( \frac{p}{N} \), where \( p \) is the number of sequences, which if chosen by the mechanical process, leads to a small prize, and \( N \) is the total number of all possible sequences that the mechanical process can choose.]

**Solution:** To choose a sequence that differs in exactly one position from a given \( n \) digit number, we can first choose the position where it differs (\( n \) ways), and then choose a digit for that position which is different from the original digit (9 ways). Thus there are \( p = 9n \) strings which, if chosen by the mechanical process, will yield a smaller prize. In all, there are \( N = 10^n \) strings. The probability of getting a smaller prize is therefore \( \frac{9n}{10^n} \).

4. **Sorted Strings**

Consider strings made up of lowercase letters, \( \text{a-z} \). We say that a string is a “sorted string” if the letters in it appear in alphabetic order. For instance, \( \text{bbn} \) and \( \text{tux} \) are sorted strings, but \( \text{ibm} \) is not.

(a) How many sorted strings of length 3 are there?

[Hint: Can you relate a sorted string to a multi-set?]

**Solution:** There is a bijection between the set of all sorted strings of length 3 and the set of all multi-sets of size 3 (with elements from the set of all letters).

Hence the number of sorted strings of length 3 is equal to the number of size-3 multi-sets of letters. This is equal to the number of ways in which 3 balls can be thrown into 26 bins. By the “stars-and-bars” technique, this is equal to \( \binom{28}{3} \).
(b) How many sorted strings of length 3 are there in which no letter repeats? (Thus bbn should not be counted, but tux should be.)

Solution: There is a bijection between the set of all sorted strings of length 3 with no repetitions and the set of all sets of size 3 (with elements from the set of all letters).

Hence the number of sorted strings of length 3 is equal to the number of size-3 sets of letters. This is simply \( \binom{26}{3} \).