1. Below is a proof by strong induction that, for any \( n \in \mathbb{Z}^+ \), any connected graph with \( n \) nodes has at least \( n - 1 \) edges. The induction variable is \( n \).

Fill in the blanks below to complete the proof. \([25 \text{ points}]\)

(a) The base case claim, which is clearly true, is that every connected graph with 1 node has at least 0 edges.

(b) The induction step. We claim that:

\[ \forall k \geq 2, \text{ if } \exists n \in \mathbb{Z}^+ \text{ such that } n \leq k - 1, \text{ it holds that any connected graph } G \text{ with } n \text{ nodes has at least } n - 1 \text{ edges,} \]

(to prove:) then any connected graph \( G \) with \( k \) nodes has at least \( k - 1 \) edges.

(c) To prove the above claim, consider an arbitrary connected graph \( G \) with \( k \) nodes. Let \( m \) denote the number of edges in \( G \). We need to prove that \( m \geq k - 1 \).

Let \( u \) be an arbitrary node in \( G \). Let \( d \) be the degree of \( u \). From \( G \), if we remove \( u \) and the \( d \) edges connected to it, we obtain a subgraph \( H \) of \( G \).

Let \( t \) denote the number of connected components in \( H \). Then (give an upper bound)

\[ t \leq d \] \hspace{1cm} (1)

(Justification omitted.)
For each $i = 1, \ldots, t$, let $n_i$ denote the number of vertices in the $i^{th}$ connected component of $H$, and $m_i$ be the number of edges in it. Clearly,

$$\sum_{i=1}^{t} n_i = k - 1 \quad (2)$$

Also, for each $i = 1, \ldots, t$, since $n_i \leq k - 1$, by the induction hypothesis (relate $n_i$ and $m_i$):

$$m_i \geq n_i - 1 \quad (3)$$

Finally, the number of edges in $G$, $m = d + \sum_{i=1}^{t} m_i$ (relate it to $m_i$). Hence, by equations (1), (2) and (3), (complete the proof)

$$m \geq d + \sum_{i=1}^{t} (n_i - 1)$$

$$= d + (k - 1) - t \geq k - 1$$.
1. Below is a proof by strong induction that, for any $n \in \mathbb{Z}^+$, any connected graph with $n$ nodes has at least $n - 1$ edges. The induction variable is $n$.

Fill in the blanks below to complete the proof. [25 points]

(a) The base case claim, which is clearly true, is that _______

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 every connected graph with 1 node has at least 0 edges _______.

(b) The induction step. We claim that:

\[
\begin{align*}
\text{(range for } k : & ) \quad \forall k \geq 1, \text{ if } \\
\text{(induction hypothesis: } & \forall n \in \mathbb{Z}^+ \text{ such that } n \leq k, \text{ it holds that any connected graph } G \\
& \text{ with } n \text{ nodes has } \underline{\text{at least } n - 1} \text{ edges,} \\
\text{(to prove: ) } & \text{then any connected graph } G \text{ with } k + 1 \text{ nodes has at least } k \text{ edges.}
\end{align*}
\]

(c) To prove the above claim, consider an arbitrary connected graph $G$ with ________ nodes. Let $m$ denote the number of edges in $G$. We need to prove that $m \underline{\geq k}$.

Let $u$ be an arbitrary node in $G$. Let $d$ be the degree of $u$. From $G$, if we remove $u$ and the $d$ edges connected to it, we obtain a subgraph $H$ of $G$.

Let $t$ denote the number of connected components in $H$. Then (give an upper bound)

\[
t \underline{\leq d}
\]

(Justification omitted.)
For each $i = 1, \ldots, t$, let $n_i$ denote the number of vertices in the $i^{th}$ connected component of $H$, and $m_i$ be the number of edges in it. Clearly,

$$
\sum_{i=1}^{t} n_i = k \tag{2}
$$

Also, for each $i = 1, \ldots, t$, since $n_i \leq k$, by the induction hypothesis (relate $n_i$ and $m_i$):

$$
m_i \geq n_i - 1 \tag{3}
$$

Finally, the number of edges in $G$, $m = d + \sum_{i=1}^{t} m_i$ (relate it to $m_i$). Hence, by equations (1), (2) and (3), (complete the proof)

$$
m \geq d + \sum_{i=1}^{t} (n_i - 1) \\
= d + k - t \geq k
$$