1. **Induction** [12 points] Prove the following claim by induction. You must write your inductive hypothesis as strong, i.e. true for all (not just one) smaller value of $n$.

Claim: $\sum_{k=1}^{n}(8k - 5) = 4n^2 - n$, for any positive integer $n$,

**Solution:** Define $P(n)$ to be $\sum_{k=1}^{n} 8k - 5 = 4n^2 - n$.

Step1 If $n = 1$ then $\sum_{k=1}^{n} 8k - 5 = 3$ and also $4n^2 - n = 3$. So $P(1)$ is true.

Step2 Now, suppose that $P(k)$ is true for any natural number $k=1,2,...n$. That is $\sum_{i=1}^{k} (8i - 5) = 4k^2 - k$ for any natural number $k=1,2,...n$.

Then

$$\sum_{i=1}^{n+1} (8i - 5) = \sum_{i=1}^{n} (8i - 5) + [8(n + 1) - 5]$$

$$= 4n^2 - n + [8(n + 1) - 5]$$

$$= 4n^2 - n + 8n + 3$$

$$= 4n^2 + 8n - n + 4 - 1 = 4n^2 + 8n + 4 - (n + 1)$$

$$= 4(n + 1)^2 - (n + 1)$$

So $P(n + 1)$ is true i.e. $\sum_{i=1}^{n+1} (8i - 5) = 4(n + 1)^2 - (n + 1)$ which is what we needed to show.

2. **Graphs** [14 points]

Claim: an edge $e$ in a graph is a cut edge if and only if $e$ is not part of a cycle.

(a) (10 points) Prove the backwards direction of this claim by contrapositive. That is, prove the contrapositive of the following: if an edge $e$ in a graph is not part of a cycle then $e$ is a cut edge.

(b) (4 points) Find all the cut edges in the graph below. Use the claim to explain briefly why none of the other edges can be cut edges.
Hint 1: graph terminology varies. Check the definitions of the terms in the textbook.

Hint 2: you may wish to use the following alternative definition of “cut edge”: An edge e is a cut edge iff there is a pair of distinct nodes p and q such that every path from p to q goes through the edge e.

Solution:

a) By contrapositive suppose e is not a cut edge then we will show that e is a part of cycle. Since e is not a cut edge then any two nodes connecting by e should have another path which does not pass through e. Say we have nodes a,b connecting with c and another path not including e. Let’s take the node a and the path not containing e. If we union this path with c we get a cycle starting and ending at a. So e is a part of a cycle.

b) Cut edges are (I,J),(J,F),(H,G),(E,A). None of others can be a cut edge since they all are part of a cycle and claim says if an edge is a part of cycle then it can not be an edge cut.

3. Scheduling Final Exams [14 points] The Department of Mathematics wants to schedule the final exams so that no student has two exams at the same day. What is the minimum number of days required to schedule exams for the following situation?

- There are eight courses a,b,c,d,e,f,g,h
- Fred is taking a,f,c
- Harry is taking b,h,f
- Hermione is taking b,c,d,f
- Ginny is taking a,h,b,c
- George is taking h,d,e
• Ron is taking h,g,f
• Percy is taking a,d

Solve this problem by building and coloring a graph representation of the problem. Specifically

(a) (3 points) For each node, give a list of nodes it is directly connected to. Each list must be in alphabetical order.

(b) (2 points) Find the largest subgraph H that is a complete subgraph (aka copy of $K_n$ for some $n$). Give a list of the nodes in H.

(c) (3 points) Draw a picture of the part of the graph that isn’t in H. I.e. show the edges that aren’t in H, along with the endpoints of these edges.

(d) (6 points) Find the chromatic number of the graph and briefly justify why your answer is correct.

Hint 1: Define the classes as nodes and build an edge between two nodes if the corresponding courses share a student.

Hint 2: Justifying a chromatic number typically involves giving a coloring for the graph and also explaining why it cannot be colored with fewer colors.

Solution:

(a) a is connected directly to b,c,d,f,h
   b is connected directly to a,c,d,f,h
   c is connected directly to a,b,d,f,h
   d is connected directly to a,b,c,e,f,h
   e is connected directly to d,h
   f is connected directly to a,b,c,d,g,h
   g is connected directly to f,h
   h is connected directly to a,b,c,d,e,f,g

Figure 2:
(b) Figure 3 is a picture of the whole graph and the complete subgraph. The list of nodes in this subgraph (what you were actually required to submit) is: a,b,c,d,f,h.

Figure 3:

(c) Here is a picture of the edges not in the complete subgraph:

Figure 4:

(d) i) The chromatic number of a graph is always greater than the chromatic number of any of its subgraphs. By part a we see that the original graph has a complete subgraph with 6 nodes. The chromatic number of any complete graph is equal to the number of nodes. Hence we can conclude that $\chi(G) \geq 6$.

ii) We can color the complete subgraph with any 6 colors, e.g. a red, b green, c blue, d yellow, f black, h purple. Since g is not directly connected to c, I can give g the same color (blue) as c. Also since e is not directly connected to b, I can give e the same color (green) as b. Hence I can color my graph with at least 6 different colors which means $\chi(G) \leq 6$.

By i) and ii), we conclude that $6 \leq \chi(G) \leq 6 \Rightarrow \chi(G) = 6$