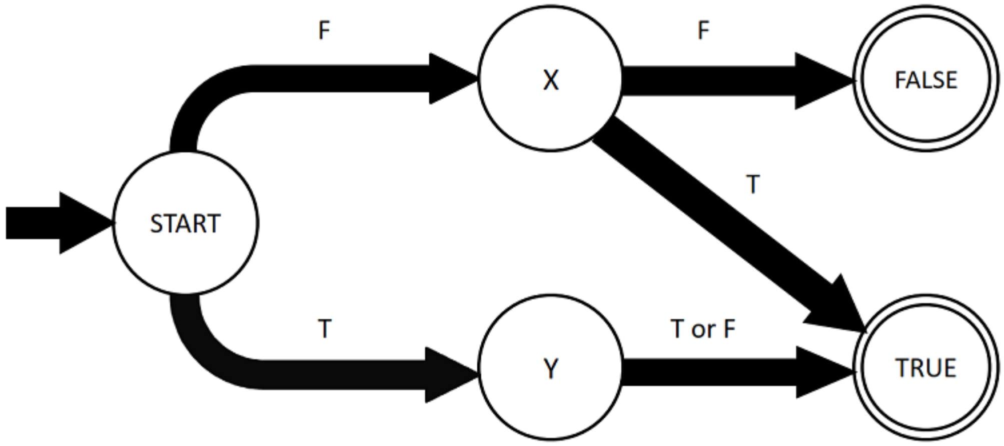
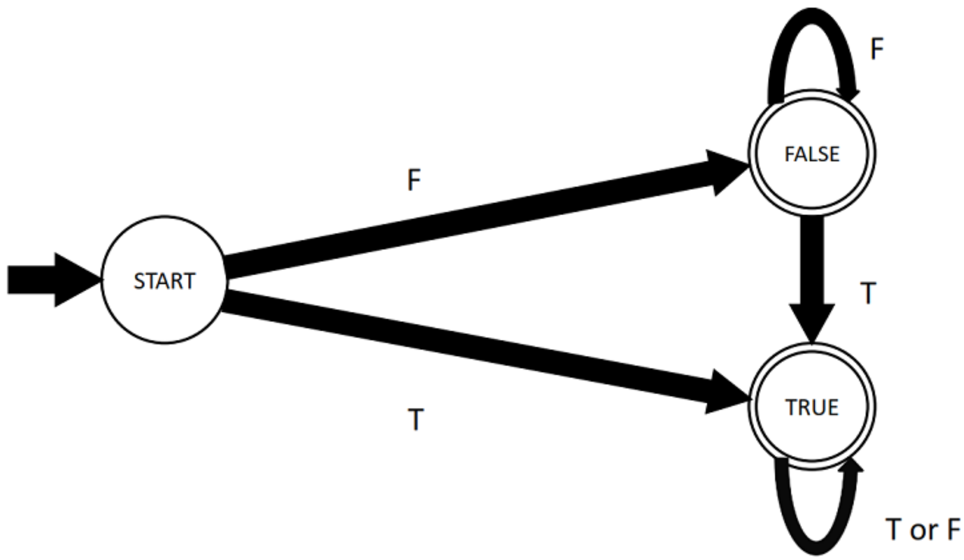
CS 173: Discrete Structures, Spring 2013
Homework 10

1. (12 points)



(a) (6 points) Maxwell Smart was told to design a machine that accepts strings $x_1 x_2 \ldots x_n$ for $x_i \in \{F, T\}$ with $(((x_1 \vee x_2) \vee x_3) \vee \ldots) \vee x_n$ true. However, the design he produced (above) only reads in strings containing exactly two characters. Create a better finite-state machine that can read in an arbitrary number of characters. Your machine should have only three states. You can use the figure below to start with.

Solution



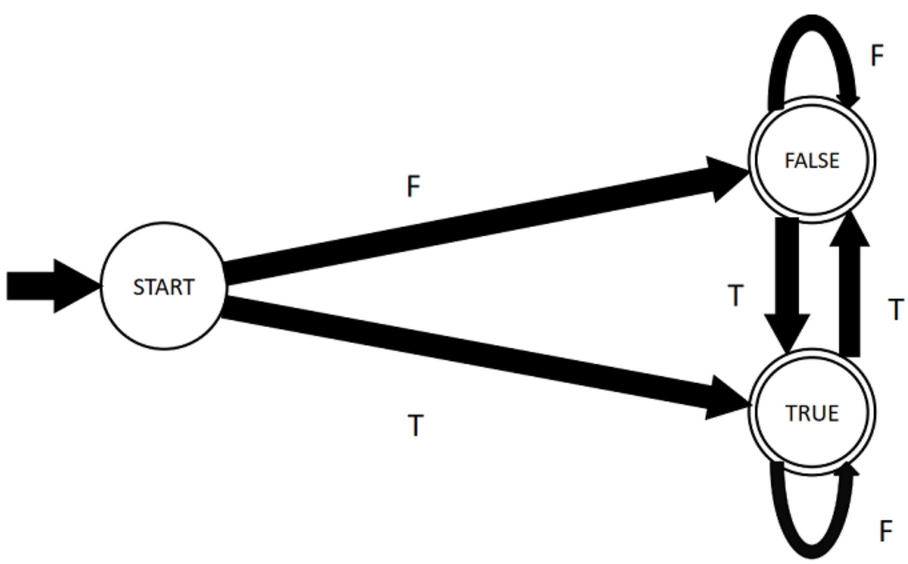
(b) (6 points) Describe a similar finite-state machine, which only accepts strings, $x_1x_2x_3...x_n$ for $x_i \in \{F, T\}$, with $(((x_1 \oplus x_2) \oplus x_3) \oplus ...) \oplus x_n$ true.

Recall that $a \oplus b$ is given by

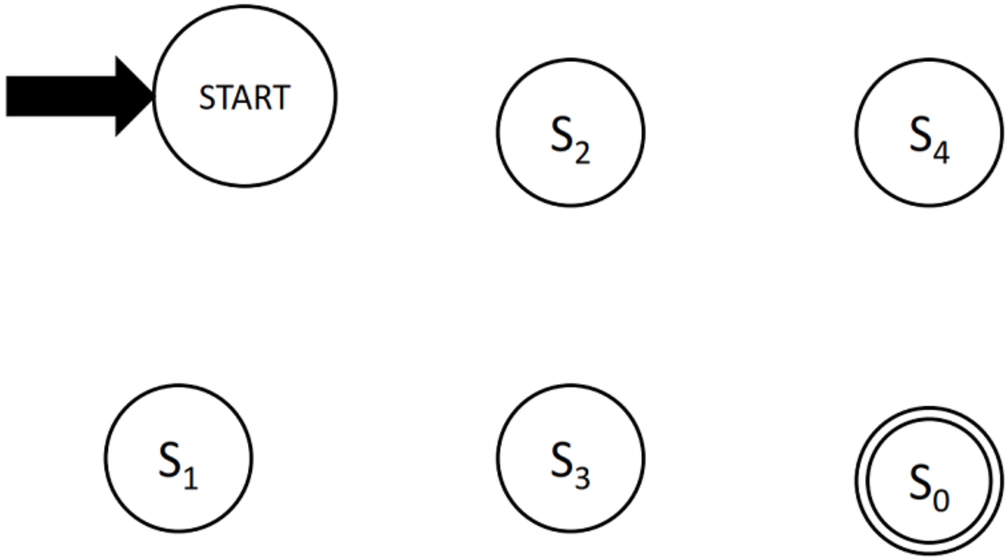
a	b	a ⊕ b
F	F	F
F	T	T
T	F	T
T	T	F

.

Solution



2. (12 points, 2 bonus) Suppose a player repeatedly flips a coin, which has an equal chance of being "heads" or "tails". If heads, the player gets \$1, if tails, the player gets \$2. The player must stop when the total is a non-zero multiple of \$5.



(a) (6 points) Finish the state diagram for this game, starting from the figure above. You'll need to add in the transitions, which should be labeled "H" (heads) or "T" (tails). Here, S_i represents the state of the player having received some amount of money congruent to i $(mod\ 5)$.

(b) (2 points) What are all of the unique sequences such that the game ends in exactly five flips?

Your sequence should be a series of H's and T's. For example, $HHTTT$ indicates two head flips, followed by three tail flips.

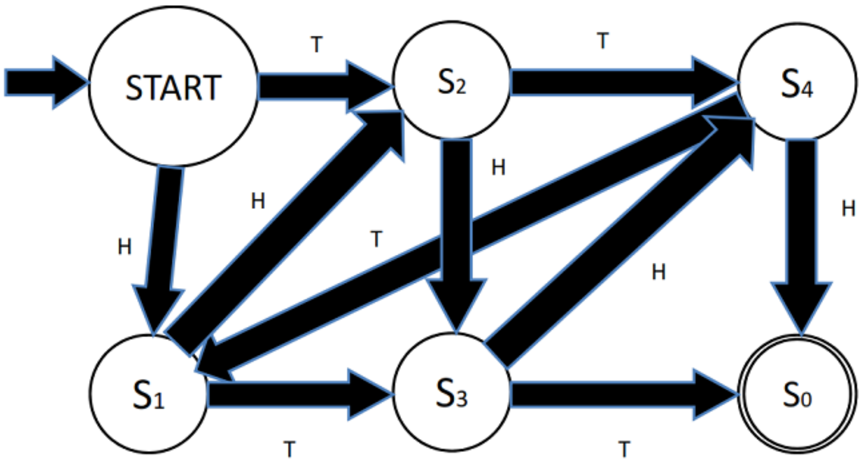
(c) (2 points) How many sequences go from start to S_0 without containing the S_4, S_1 subsequence? For example, $START, S_1, S_3, S_4, S_0$ is one such sequence.

(d) (2 points) How many sequences go from start to S_0 containing exactly one S_4, S_1 subsequence? For example, $START, S_1, S_3, S_4, S_1, S_3, S_0$ is one such sequence.

(e) (bonus 2 points) If the player pays \$7 to play each game, and plays many games, should the player expect to make money or lose money? Briefly justify or explain how you got your answer. Provide an approximate (or exact) value for the expected payout. You could solve this by explicitly computing or by writing a program to estimate the expected payout through many simulations.

Note that the expected payout is $5 \cdot Pr(5) + 10 \cdot Pr(10) + 10 \cdot Pr(15) + ...$, where $Pr(x)$ is the probability of ending the game at exactly x dollars.

Solutions



(a)

(b) To end the game at exactly five flips, one can do $HHHHH$ or $TTTTT$.

(c) The paths that go from $START$ to S_0, without the S_4, S_1 subsequence, are five heads, three heads and one tails, and one heads and two tails. There is $\binom{5}{5} = 1$ path with exactly five heads. There are $\binom{4}{3}$ paths with three heads and one tails. There are $\binom{3}{1}$ paths with one heads and two tails. In total, there are $1 + \binom{4}{3} + \binom{3}{1} = 1 + 4 + 3 = 8$ paths that terminate at \$5.

(d) First we consider the paths that go from $START$ to S_1 containing exactly one S_4, S_1 subsequence. These paths consist of all paths from $START$ to S_4 with one last tails flip. They are four heads, two heads and one tails, or two tails. There is $\binom{4}{4} = 1$ path with exactly four heads. There are $\binom{3}{2}$ paths with two heads and one tails. Finally, there is $\binom{2}{0} = 1$ path with two tails. In total this is $1 + \binom{3}{2} + 1 = 1 + 3 + 1 = 5$ paths that go from $START$ to S_4. Appending a tails flip to each such path brings us to S_1.

Then, consider all paths from S_1 to S_0. These paths consist of all paths with four heads, two heads and one tails, or two tails. There is $\binom{4}{4} = 1$ path with exactly four heads. There are $\binom{3}{2}$ paths with two heads and one tails. Finally, there is $\binom{2}{0} = 1$ path with two tails. In total this is $1 + \binom{3}{2} + 1 = 1 + 3 + 1 = 5$ paths that go from S_1 to S_0.

Considering that any path going from $START$ to S_0 with exactly one S_4, S_1 subsequence must contain one of each of these paths, we conclude there are $5 \cdot 5 = 25$ such paths.

(e) We start by computing the probability that a random walk at $START$ will end up at S_0 without the S_4, S_1 subsequence. This is $Pr(5) = 0.6563$.

Next we compute the probability that a random walk at S_1 will end up at S_0 without the S_4, S_1 subsequence. This is $\alpha = 0.6875$.

Given these, we know that the probability of a walk at $START$ ending up at S_1 using exactly one S_4, S_1 subsequence is $1 - Pr(5) = 0.3437$. This is because such a walk must skip S_0 by using one S_4, S_1 subsequence. Alternately, this is the probability that a walk starting at $START$ will end up at S_4, and then take the "T" transition to S_1.

Also, the probability of a walk starting at S_1 ending up at S_1 again, with exactly one S_4, S_1 subsequence is $1 - \alpha = 0.3125$, for similar reasons.

Then, $Pr(10) = (1 - Pr(5))\alpha$, because any path ending at \$10 must take one S_4, S_1 subsequence (this is the $1 - Pr(5)$), and then go from S_1 to S_0 (this is the α). Similarly, $Pr(15) = (1 - Pr(5))(1 - \alpha)\alpha$, $Pr(20) = (1 - Pr(5))(1 - \alpha)^2\alpha$, etc.

Then, we get the expected payout is $Pr(5) + \sum_{i=0}^{\infty}(1 - Pr(5))(1 - \alpha)^i\alpha(5*(i+2)) \approx 7.5$. Thus a player should expect to gain money.