CS 173: Discrete Structures, Spring 2012
Homework 5

This homework contains 4 problems worth a total of 50 points.
It is due on Wednesday, Feb. 29th at 3pm.

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1. **[12 points] Pigeonhole Principle**

   Prove a pair of the three Pigeonhole Principle problems presented.

   [Perpetual Hint: pair means (at least) two (distinct).]

   (a) Prove that there exist a pair of powers of 5 whose difference is divisible by 2012.
   [Note: The fifth month is May which is a substring of Mayan which is a substring of "Mayan 2012 Doomsday". woo]

   Sol: Let our pigeons be the infinite set of powers of 5, we can map these to the holes $\mathbb{Z}_{2012}$, by their congruence class modulo 2012, there are only $2012 (< \infty)$ such classes. Then by the Pigeonhole Principle there are two powers of 5 in the same congruence class, hence 2012 divides their difference, by the definition of modular congruence.
   FYI: all pairs are of the form $5^j, 5^{j+502k}$ where $j \in \mathbb{N}$ and $k \in \mathbb{Z}^+$.
   The note is to get get you to realize there is nothing special about these numbers, that is replace 5 with $n \in \mathbb{Z} - \{-1, 0, 1\}$ and 2012 with $m \in \mathbb{Z}^+$ and it still works.

   (b) Twenty three people go to lunch at Lai Lai, a Chinese restaurant on Green Street, and sit down evenly spaced at a large circular table with a lazy susan (central rotating circular tray) on it, each of them orders a different dish from the menu and they all refuse to share. The dishes of food are brought out and placed on the lazy susan, one in front of every person, but they are entirely mismatched, so each person has another person's dish. Prove that there is a way to rotate the lazy susan so that at least two people have the correct dish that they ordered in front of them.

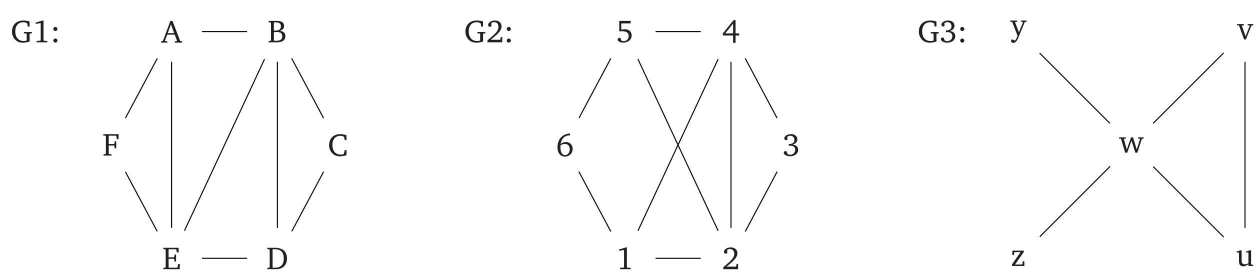
   Sol: Let our pigeons be our 23 people, and to each of these people we can assign a number in $\{1, 2, \cdots, 22\}$, representing how many times the lazy susan needs to be rotated clockwise (incrementing one person at a time) in order to align that person with their dish. Note this is never 0, since the dishes were completely mismatched. Then by the Pigeonhole Principle there are at least two people whose dish is the same rotation away, since $23 > 22$.

   (c) There are n people in a room. Consider the various acquaintances amongst them, these form the edges of a simple graph with the nodes being the people. [So the adjacency/acquaintance relation is symmetric and irreflexive.] Show that if $n \geq 2$ then some pair of them must have the same number of acquaintances.

   Sol: Let our pigeons be our n people, and to each of these people we will assign their number of acquaintances. There are two cases: someone is acquainted with everyone, so each person's number of acquaintances is between 1 and $n-1$; or no one is acquainted with everyone, so each person's number of acquaintances is between 0

and $n-2$. In either case there are at most $n-1$ different numbers for each of the n people to map into. By the Pigeonhole Principle there are at least two people who have the same number of acquaintances.

2. **[14 points] Graph Isomorphisms**



(a) Prove there is no graph isomorphism between the graphs $G1$ and $G2$.

Sol: The number of nodes in each is 6, the number of edges in each is 9, the list of degrees of the nodes in each is $[2,2,3,3,4,4]$. So we cannot conclude anything so far, and are forced to look at the adjacency requirements of a graph isomorphism, or the number of subgraphs of certain types. Any one of the following will work:

- In $G1$ the degree 4 nodes have no common adjacent degree 2 node, but in $G2$ the degree 4 nodes have a common degree 2 node, namely the node labeled '3'.
- In $G1$ the degree 3 nodes have no common adjacent degree 2 node, but in $G2$ the degree 3 nodes have a common degree 2 node, namely the node labeled '6'.
- In $G1$ each of the two degree 2 nodes is adjacent to both a degree 3 node and a degree 4 node, but in $G2$ the degree 2 node '6' is only adjacent to degree 3 nodes and the degree 2 node '3' is only adjacent to degree 4 nodes.
- There are four C_3's in $G1$, but only three C_3's in $G2$. ♠
- There are four C_4's in $G2$, but only three C_4's in $G1$. ♠
- There are twenty four graph morphisms from C_3 to $G1$, but only eighteen graph morphisms from C_3 to $G2$. ♡
- There are thirty two graph morphisms from C_4 to $G2$, but only twenty four graph morphisms from C_4 to $G1$. ♡

  ♠ ignoring starting point and direction traversed

  ♡ graph morphism=graph isomorphism to a subgraph, this keeps track of starting point and direction traversed

(b) Count the number of different graph isomorphisms from $G3$ to itself.

Sol: For any graph isomorphism: degrees must match up in the bijection between the nodes. This forces: w can only map to w (the only degree 4 node), $\{y,z\}$ can only map to $\{y,z\}$ (degree 1 nodes), and $\{u,v\}$ can only map to $\{u,v\}$ (degree 2 nodes). With these restraints there are only four possible node bijections, and all <u>four</u> of these yield valid graph isomorphisms, namely the nodes $yzwuv$ map to $yzwuv$, $yzwvu$, $zywuv$, or $zywvu$, respectively.

3. **[12 points] Function proofs**

For each of the following functions:

(i) briefly prove or disprove it is one-to-one (1-1), and

(ii) briefly prove or disprove it is onto:

(a) $f : \{3, 5, -117\} \times \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Q} - \{0\}$ such that $f(a, b, c) = ab/c$

Not 1-1: $f(3, 1, 1) = 3 \cdot 1/1 = 3 = 3 \cdot 2/2 = f(3, 2, 2)$ and $(3, 1, 1) \neq (3, 2, 2)$,
or $f(3, 5, 1) = 3 \cdot 5/1 = 15 = 5 \cdot 3/1 = f(5, 3, 1)$ and $(3, 5, 1) \neq (5, 3, 1)$, so not 1-1.
Yes Onto: here is how to hit any $\frac{m}{n} \in \mathbb{Q} - \{0\}$, where $m \in \mathbb{Z} - \{0\},\ n \in \mathbb{Z}^+$:
if $m < 0$ then $f(-117, -m, 117 \cdot n) = -117 \cdot -m/(117 \cdot n) = m/n$,
else $m > 0$ then $f(3, m, 3 \cdot n) = 3 \cdot m/(3 \cdot n) = m/n$.
So all of $\mathbb{Q} - \{0\}$ is in the image of f, hence it is onto.

(b) $g : I \times \mathbb{N} \longrightarrow \mathbb{R}$ such that $g(x, y) = x + y$ where I is $[0, 1) = \{x \in \mathbb{R} \mid 0 \leq x \wedge x < 1\}$

Yes 1-1: if $g(x, y) = g(a, b) \Rightarrow x + y = a + b$, now using $y, b \in \mathbb{N}$ and a property of floor
$\Rightarrow y = \lfloor x \rfloor + y = \lfloor x + y \rfloor = \lfloor a + b \rfloor = \lfloor a \rfloor + b = b \Rightarrow x = a \Rightarrow (x, y) = (a, b)$ so g is 1-1.
Not Onto: $-1 \in \mathbb{R}$ but $g(x, y) = x + y \geq 0$ so -1 is not in the image, hence not onto.

(c) $h : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $h(u, v) = \lfloor u \rfloor + 2^v$

Not 1-1: $h(0, 0) = 0 + 2^0 = 1 = 0 + 2^0 = h(0.5, 0)$ and $(0, 0) \neq (0.5, 0)$, so not 1-1.
Yes Onto: here is how to hit any $z \in \mathbb{R}$. we can do one all encompassing case:
$h(z - 1, \log_2 (z - \lfloor z \rfloor + 1)) = \lfloor z - 1 \rfloor + 2^{\log_2 (z - \lfloor z \rfloor + 1)} = \lfloor z \rfloor - 1 + z - \lfloor z \rfloor + 1 = z.$
or break it into two cases: if $z \in \mathbb{Z}$ then $h(z - 1, 0) = \lfloor z - 1 \rfloor + 2^0 = z - 1 + 1 = z$
else $z \in \mathbb{R} - \mathbb{Z}$ so let $a = z - \lfloor z \rfloor > 0$ then $h(z, \log_2 a) = \lfloor z \rfloor + 2^{\log_2 a} = \lfloor z \rfloor + z - \lfloor z \rfloor = z$.
So all of $\mathbb{R}$ is in the image of h, hence it is onto.

4. **[12 points] Bijection application**

Let L be a set of letters $\{a, b, c, d, e, f, g, h\}$, of cardinality $n = 8$, and let S be the set of all strings(ordered lists) of letters in L of length n, where each letter appears exactly once in the string. So $c, abba \notin S$, but $cgehbadf \in S$ along with all the $n! = 8!$ permutations of it.

We can view a permutation of the set $I = \{1, 2, \cdots, n\}$ as a set of instructions for taking any string in S and reordering its letters to get a new string in S. For example, the permutation 54817326 per**mutates** the string $s = cgehbadf$ into $bhfcdega$. Here is how it works: write the 5^{th} letter of s (b), followed by the 4^{th} letter (h), then the 8^{th} letter (f), $\cdots$ etc., and finally the 6^{th} letter (a). We will denote this permuting function by $\rho_{54817326} : S \longrightarrow S$, and so we write $\rho_{54817326}(cgehbadf) = bhfcdega$. In general $\rho_{i_1 i_2 \cdots i_n}(\ell_1 \ell_2 \cdots \ell_n) = \ell_{i_1} \ell_{i_2} \cdots \ell_{i_n}$, that is the new output string's j^{th} letter is the i_j^{th} letter of the input string.

Lets reverse this procedure and start with two strings and ask how one is permutated to form the other. To figure out what permutation takes $s = cgehbadf$ to $t = haebfgdc$: we ask in which spot does h occur in s (in the 4^{th} spot) then a (the 6^{th} spot) then e $\cdots$ etc. to get $\rho_{46358271}(cgehbadf) = haebfgdc$.

Now lets compose two permutations $\rho_{72638145}(\rho_{54817326}(cgehbadf)) = \rho_{72638145}(bhfcdega) = ghefabcd$. Using the reversing procedure above we find that 24386517 is the unique permutation which takes that original input to that final output, that is $\rho_{24386517}(cgehbadf) =$

$ghefabcd$, and in fact $\rho_{24386517}(t) = \rho_{72638145}(\rho_{54817326}(t))$ for *every* $t \in S$ so we conclude $\rho_{24386517} = \rho_{72638145} \circ \rho_{54817326}$. The composition of two permutations is a permutation.

Furthermore every permutation has an inverse permutation, meaning when you compose it with its inverse, in either order, you get the identity permutation. Eg $\rho_{54817326} \circ \rho_{47621853} = \rho_{47621853} \circ \rho_{54817326} = \rho_{12345678}$ so $\rho_{54817326}$ and $\rho_{47621853}$ are inverses to each other.

(a) Which permutation $k_1k_2\cdots k_n$ satisfies $gcfbdaeh = \rho_{k_1k_2\cdots k_n}(fabgdhec)$?

Sol: $\begin{array}{c|c} 48135276 & 12345678 \\ gcfbdaeh & fabgdhec \end{array}$ so $k_1k_2\cdots k_n = 48135276$

(b) Compute $\rho_{72638145}(fabgdhec)$ and $\rho_{71328564}(\rho_{72638145}(fabgdhec))$.

Sol: $\begin{array}{c|c} 72638145 & 12345678 \\ eahbcfgd & fabgdhec \end{array}$ so $\rho_{72638145}(fabgdhec) = eahbcfgd$

$\begin{array}{c|c} 71328564 & 12345678 \\ gehadcfb & eahbcfgd \end{array}$ so $\rho_{71328564}(eahbcfgd) = gehadcfb$

(c) What permutation $\rho_{m_1m_2\cdots m_n}$ corresponds to $\rho_{71328564} \circ \rho_{72638145}$?

Sol: by (b) $\rho_{71328564}(\rho_{72638145}(fabgdhec)) = gehadcfb$
so the question is to find $\rho_{????????}(fabgdhec) = gehadcfb$

$\begin{array}{c|c} 47625813 & 12345678 \\ gehadcfb & fabgdhec \end{array}$ so $\rho_{71328564} \circ \rho_{72638145} = \rho_{47625813}$

(d) What is the inverse permutation of $\rho_{72638145}$?

Sol: by (b) $\rho_{72638145}(fabgdhec) = eahbcfgd$
so the question is to find $fabgdhec = \rho_{????????}(eahbcfgd)$

$\begin{array}{c|c} 62478315 & 12345678 \\ fabgdhec & eahbcfgd \end{array}$ so ${\rho_{72638145}}^{-1} = \rho_{62478315}$