Algorithms and Trees

Check the box that best characterizes each item.

\[
\sum_{k=0}^{n-1} 2^k
\]

2^n - 2: \quad \square \quad 2^n - 1: \quad \checkmark

2^{n-1} - 1: \quad \square

The level of the root node in a tree of height \( h \).

0: \quad \checkmark

1: \quad \square

\( h - 1 \): \quad \square

\( h \): \quad \square

\( h + 1 \): \quad \square

How often is the root node of a tree an internal node? never: \quad \square

sometimes: \quad \checkmark

always: \quad \square

Short answer

(a) Suppose that \( g: A \to B \) and \( f: B \to C \). Prof. Snape claims that if \( f \circ g \) is onto, then \( g \) is onto. Disprove this claim using a concrete counter-example in which \( A, B, \) and \( C \) are all small finite sets.

Solution: Suppose that \( A = \{1, 2\}, B = \{3, 4, 5\}, \) and \( C = \{\text{red, blue}\} \). Define \( g \) by \( g(1) = 3 \) and \( g(2) = 5 \). Define \( f \) by \( f(3) = \text{red}, f(4) = \text{red}, \) and \( f(5) = \text{blue} \). Then \( (f \circ g)(1) = \text{red} \) and \( (f \circ g)(2) = \text{blue} \). So \( f \circ g \) is onto because every element of \( C \) has a pre-image. However, \( g \) isn’t onto because no element of \( A \) maps onto 4.

(b) Suppose that \( A, B \) and \( C \) are sets. Recall the definition of \( X \subseteq Y \): for every \( p \), if \( p \in X \), then \( p \in Y \). Prove that if \( A \subseteq B \) then \( A \cap C \subseteq B \cap C \). Briefly justify the key steps in your proof.

Solution: Suppose that \( p \in A \cap C \). Then \( p \in A \) and \( p \in C \), by the definition of intersection. Since \( p \in A \) and \( A \subseteq B \), \( p \in B \) (definition of subset). So \( p \in B \) and \( p \in C \), which implies that \( p \in B \cap C \) (definition of intersection).

(c) Suppose that \( g: \mathbb{Z} \to \mathbb{Z} \) is one-to-one. Let’s define the function \( f: \mathbb{Z} \to \mathbb{Z}^2 \) by \( f(x) = (x^2, g(x)) \). Prove that \( f \) is one-to-one.

Solution: Let \( x \) and \( y \) be integers. Suppose that \( f(x) = f(y) \). By the definition of \( f \), this means that \( (x^2, g(x)) = (y^2, g(y)) \). So then \( x^2 = y^2 \) and \( g(x) = g(y) \). Since \( g(x) = g(y) \) and \( g \) is one-to-one, \( x = y \).
So we have that $f(x) = f(y)$ implies $x = y$. This means that $f$ is one-to-one.

(d) How many different 6-letter strings can I make out of the letters in the word “illini”?

**Solution:** We calculate the number of permutations of 6 letters (6!) and divide out by the double-counting of the possibilities for l (2!) and for i (3!). This gives us $\frac{6!}{2!3!} = 5 \cdot 4 \cdot 3 = 60$ possible strings.

(e) Define the function $f$ as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n + 1) = 5f(n) - 6f(n - 1)$

Suppose we’re proving that $f(n) = 3^n - 2^n$ for every positive integer $n$. State the inductive hypothesis and the conclusion of the inductive step.

**Solution:** Inductive hypothesis: suppose that $f(n) = 3^n - 2^n$ for $n = 1, 2, \ldots k$, for some integer $k$.

Conclusion of the inductive step: $f(k + 1) = 3^{k+1} - 2^{k+1}$.

Note 1: a strong hypothesis is required because the formula reaches back two integers.

Note 2: the variable $k$ in the conclusion matches the upper bound in the hypothesis. A common mistake is to have it match the variable in the hypothesis equation ($n$). We’re assuming that the equation holds for all values up through $k$, so we need to prove it holds for $k + 1$.

**Induction**

Let the function $f : \mathbb{N} \rightarrow \mathbb{Z}$ be defined by

- $f(0) = 1$
- $f(1) = 6$
- $\forall n \geq 2, f(n) = 6f(n - 1) - 9f(n - 2)$

Use strong induction on $n$ to prove that $\forall n \geq 0, f(n) = (1 + n)3^n$.

Base case(s):

**Solution:** $f(0) = 1 = (1 + 0)3^0$ and $f(1) = 6 = (1 + 1)3^1$. We need to check two base cases because the inductive step will reach back two integers.

Inductive hypothesis [Be specific, don’t just refer to “the claim”]:

- Assume $f(k) = (1 + k)3^k$ for some integer $k$.
- Prove $f(k + 1) = (1 + k + 1)3^{k+1}$.
Solution: Suppose that \( f(n) = (1 + n)3^n \) for \( n = 0, 1, \ldots, k \), for some \( k \geq 2 \).

Rest of the inductive step:

Solution: \( f(k+1) = 6f(k) - 9f(k-1) \) by the definition of \( f \). By the inductive hypothesis, we know that \( f(k) = (1 + k)3^k \) and \( f(k - 1) = k3^{k-1} \). So by substituting, we get

\[
\begin{align*}
  f(k + 1) &= 6(1 + k)3^k - 9k3^{k-1} \\
          &= 2(1 + k)3^{k+1} - k3^{k+1} \\
          &= 2 \cdot 3^{k+1} + 2 \cdot k3^{k+1} - k3^{k+1} \\
          &= 2 \cdot 3^{k+1} + 3^{k+1} \\
          &= (k + 2)3^{k+1}
\end{align*}
\]

So \( f(k + 1) = (k + 2)3^{k+1} \), which is what we needed to show.
1. How many connected components does each graph have?
   **Solution:** G1 has two connected components. G2 and G3 each have one connected component.

2. Are graphs G1 and G2 (above) isomorphic? Briefly justify your answer.
   **Solution:** No. G2 is connected, but G1 isn’t connected. Also, G2 contains a cycle with 6 vertices, and G1 doesn’t.

3. What is the diameter of G3?
   **Solution:** 4. (It’s the number of edges on a shortest path between the two vertices furthest apart. In this case, y and either q or r.)

4. Does G3 contain an Euler circuit? Why or why not?
   **Solution:** No, it can’t contain an Euler circuit because some of the vertices (e.g. p) have odd degree.

5. Does G2 and/or G3 contain a cut edge? If so, identify which edge(s) are cut edges.
   **Solution:** G3 contains a cut edge: the edge connecting p and s. G2 does not contain a cut edge.