More induction examples

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Here are some more examples of induction, not done in lecture this term but instructive to read.

1 A proof involving functions

Consider this claim:

Claim 1 *For any non-empty sets A and B where $|A| = |B| = n$, there are exactly $n!$ bijections from A to B.*

It may help to choose two concrete sets with 3 elements and write out all 6 bijections from one to the other.

Let's prove this using induction. Again, each $P(n)$ will be a claim about a whole collection of sets A and B.

> Proof: By induction on n, i.e. the cardinality of A and B.
>
> Base: $n = 1$. Then there is exactly one bijection from A to B, mapping the single element of A to the single element of B. In this case $n! = 1$ as well.
>
> Induction: For some positive integer k, suppose that for any sets A and B with $|A| = |B| = k$, there are $k!$ bijections from A to B.

Let A and B be any non-empty two sets with cardinality $k+1$. We need to show that there are $(k+1)!$ bijections from A to B.

Pick an element x in A. (We can do this since A is not empty.)

There are $(k+1)$ ways to choose x's image $f(x)$. To complete one of these into a bijection of A onto B, we need to make images for the rest of the elements of A. That is, we need to create a bijection from $A - \{x\}$ onto $B - \{f(x)\}$. Both $A - \{x\}$ and $B - \{f(x)\}$ contain k elements. So, by the inductive hypothesis, there are $k!$ bijections from $A - \{x\}$ onto $B - \{f(x)\}$.

So, for each of the $k+1$ choices for $f(x)$, we have $k!$ ways to complete the whole bijection. This means that we have $(k+1) \cdot k! = (k+1)!$ possible bijections from A onto B,which is what we needed to show.

2 Some commentary on socks

Some flaws in inductive proofs are fairly obvious. You might be missing the base case. Your inductive step might not state its inductive hypothesis, or might not use it. Or the algebra in the middle of the inductive step might be wrong.

Here's an interesting sort of wrong proof by induction.

Claim 2 *If D is a drawer full of socks, all socks in D have the same color.*

Proof: By induction on the cardinality of D which we'll call n.

Base: $n = 0$. If the drawer is empty, then the claim is vacuously true.

$n = 1$. If D contains only one sock, clearly all socks in D have the same color.

Induction: Suppose that we have an integer $k \geq 1$ such that all sets of k socks contain only socks of the same color. Let D be a set of $k+1$ socks.

> Since $|D| \geq 2$, let's pick two socks x and y in D. Consider the sets $A = D - \{x\}$ and $B = D - \{y\}$. Each of these sets contains k socks, so all the socks in A have the same color and similarly for all the socks in B.
>
> Now, pick any sock z in $A \cap B$. z must have the same color as x because all the socks in A have the same color. But z must also have the same color as y since all socks in B have the same color. So the colors of x and y must be the same. This means that the socks in $A \cup B$ must all share the same color. But $D = A \cup B$, so all the socks in D have the same color, which is what we needed to show.

It's obvious that the claim isn't true, so there must be a bug in this proof somewhere. But it's not easy to spot. The problem step is where you pick z in $A \cap B$. We haven't checked that $A \cap B$ contains any elements to pick from. It might be the empty set. And, in fact, it is the empty set when $n = 2$, because D contains only two elements and we've removed both of them.